

KÄHLER AND SASAKIAN-EINSTEIN QUOTIENTS

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ABSTRACT. We construct symplectic and Kähler ray reduced spaces and discuss their relation with the Marsden-Weinstein (point) reduction. This Kähler reduction is well defined even when the momentum value is not totally isotropic. The compatibility of the ray reduction with the cone construction and the Boothby-Wang fibration is presented. Using the compatibility with the cone construction we provide the exact description of ray quotients of cotangent bundles. Some applications of the ray reduction to the study of conformal Hamiltonian systems are described. We also give necessary and sufficient conditions for the (ray) quotients of Kähler (Sasakian)-Einstein manifolds to be again Kähler (Sasakian)-Einstein.

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1. INTRODUCTION

In this paper we study geometric properties of Sasakian and Kähler quotients. For manifolds endowed with a Lie group G of symmetries, we construct a reduction procedure for symplectic and Kähler manifolds using the ray pre-images of the associated momentum map J . More precisely, instead of taking as in point reduction (Weinstein-Marsden reduce spaces, usually denoted by M_μ), the pre-image of a momentum value μ , we take the pre-image of $\mathbb{R}^+\mu$, the positive ray of μ . And instead of taking the quotient with respect to the isotropy group G_μ of the momentum with respect to the coadjoint action of G , we take it with respect to the kernel group of μ , a normal subgroup of G_μ . The ray reduced spaces will be denoted by $M_{\mathbb{R}^+\mu}$. We have three reasons to develop this construction.

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One is geometric: the construction of non-zero, well defined Kähler reduced spaces. Kähler point reduction is not always well defined. The problem is that the complex structure may not leave invariant the horizontal distribution of the Riemannian submersion $\pi_\mu : J^{-1}(\mu) \rightarrow M_\mu := J^{-1}(\mathbb{R}^+\mu)/G_\mu$. The solution proposed in the literature, is based on the Shifting Theorem (see Theorem 6.5.2 in [28]). More precisely, one endows the coadjoint orbit of μ , \mathcal{O}_μ with a unique up to homotheties Kähler-Einstein metric of positive Ricci curvature. This uniqueness modulo homotheties is guaranteed by the choice of an Ad^* -invariant scalar product on \mathfrak{g}^* . Then, one performs the zero reduction of the Kähler difference of the base manifold M and \mathcal{O}_μ . Unfortunately, this construction is correct only in the case of totally isotropic momentum (i.e. $G_\mu = G$). Otherwise, using the unique Kähler-Einstein form on the coadjoint orbit, instead of the Kostant-Kirillov-Souriau form makes impossible the use of the Shifting Theorem since the momentum map of the orbit will no longer be the inclusion. Even so, one could take by definition the reduced space at μ momentum to be the zero reduced space of the symplectic difference of M and \mathcal{O}_μ . But this reduced space is not canonical, in the sense that the pull-back through the quotient projection of the reduced Kähler structure is no longer the initial one. On the other hand, the ray Kähler reduction always exists and is canonical.

The second reason is that it provides invariant submanifolds for conformal Hamiltonian systems (see [22]) and consequently, the right framework for the reduction of symmetries of such systems. They are usually non-autonomous mechanical systems with friction whose integral curves preserve, in the case of symmetries, the ray pre-images of the momentum map, and not the point pre-images.

The third reason is finding necessary and sufficient conditions for quotients of Kähler (Sasakian)-Einstein manifolds to be again Kähler (Sasakian)-Einstein. Using techniques of A. Futaki (see [12], [13]), we prove that, under appropriate hypothesis, ray quotients of Kähler-Einstein manifolds remain Kähler-Einstein. We can thus construct new examples of Kähler (Sasakian)-Einstein metrics.

As examples of symplectic (Kähler) and contact (Sasakian) ray reductions we treat the case of cotangent and cosphere bundles. We show, proving a shifting type theorem that, theoretically, $(T^*Q)_{\mathbb{R}^+\mu}$ and $(S^*Q)_{\mathbb{R}^+\mu}$ are universal ray reduced spaces. Concrete examples of toric actions on spheres are also computed.

The paper is structured as follows. Section 2 presents the symplectic and Kähler ray reduction treating separately the case of exact symplectic manifolds. In the fourth section of this paper we deal with the cone and Boothby-Wang compatibilities with the ray reduction. We show that the ray reduction of the cone of a contact manifold is exactly the cone of the contact reduced space. As a corollary we obtain the ray reduction of cotangent bundles. Also, we prove that the Boothby-Wang fibration associated to a quasi-regular, compact, Sasakian manifold descends to a Boothby-Wang fibration of the ray reduced spaces. Section 4 presents the study of conformal Hamiltonian systems. We extend the class of conformal Hamiltonian systems already studied in the literature and we complete the existing Lie Poisson reduction with the general ray one, making thus use of the conservative properties of the momentum map. We illustrate all these with the example of a certain type of Rayleigh systems. We also give a characterization of relative equilibria for this type of systems. In Section 5 we perform the ray reduction of cotangent

bundles of Lie groups, as well as the reduction of their associated cosphere bundles. We show, proving a shifting type theorem that, theoretically, $(T^*G)_{\mathbb{R}^+\mu}$ and $(S^*G)_{\mathbb{R}^+\mu}$ are universal ray reduced spaces. The role of the coadjoint orbit of the ray momentum $\mathbb{R}^+\mu$ in the construction of these universal reduced spaces is made clear. In the last section we find necessary and sufficient conditions for the ray reduced space of a Kähler-Einstein manifold of positive Ricci curvature to be again Kähler-Einstein. Using the compatibility of ray reduction with the Boothby-Wang fibration, we obtain as a corollary similar conditions for the Sasakian-Einstein case. All these are illustrated with concrete examples in which we construct new Kähler (Sasakian)-Einstein manifolds.

2. SYMPLECTIC AND KÄHLER RAY-REDUCTIONS

Let G be a Lie group acting smoothly, properly, by symplectomorphisms and in a Hamiltonian way on a symplectic manifold (M, ω) . Denote by $J : M \rightarrow \mathfrak{g}^*$ the associated momentum map and recall that it is G -equivariant. For any element $\mu \in \mathfrak{g}^*$, let K_μ be the unique connected, normal Lie subgroup of G_μ with Lie algebra given by $\mathfrak{k}_\mu = \ker(\mu|_{\mathfrak{g}_\mu})$. This group is called the *kernel group of μ* .

Definition 2.1. We define the quotient of M by G at $\mathbb{R}^+\mu$ to be $M_{\mathbb{R}^+\mu} := J^{-1}(\mathbb{R}^+\mu)/K_\mu$. $M_{\mathbb{R}^+\mu}$ will be called the *ray reduced space at μ* .

In a paper of Guillemin and Sterneberg ([14], Example 4) we found a geometric interpretation for the kernel algebra of μ . Let $\mathcal{O}_{\mathbb{R}^+\mu}$ be the *cone of the coadjoint orbit through μ* defined by

$$(2.1) \quad \mathcal{O}_{\mathbb{R}^+\mu} := \{Ad_{g^{-1}}^* r\mu \mid g \in G, r \in \mathbb{R}^+\}.$$

The conormal space at μ of $\mathcal{O}_{\mathbb{R}^+\mu}$ is precisely \mathfrak{k}_μ . This can be easily deduced using the characterization of the tangent space at μ of $\mathcal{O}_{\mathbb{R}^+\mu}$ given in Proposition 5.3.

In this section we will show that, under certain hypothesis, the ray quotient admits a natural symplectic or Kähler structure, once the initial manifold is symplectic or Kähler. The proof of the next theorem is an analogous of the proof given in [36] for the contact case (see Theorem 1). As all reduction theorems, it mainly uses arguments in linear symplectic or contact algebra.

For the two results of this section we will need three lemmas. The first is a characterization of a locally free action and the last two are classical results of symplectic linear algebra.

Lemma 2.1. *J is transverse to $\mathbb{R}^+\mu$ if and only if K_μ acts locally freely on $J^{-1}(\mathbb{R}^+\mu)$.*

Lemma 2.2. *Consider a symplectic vector space (V, Ω) and $W \subset V$ an isotropic subspace. Then, $\ker \Omega|_{W^\Omega} = W$, where W^Ω is the symplectic perpendicular of W .*

Lemma 2.3. *Let V be a vector space and $\Omega : V \times V \rightarrow \mathbb{R}$ an antisymmetric and bilinear two-form. If V admits the direct decomposition $V = X \oplus V$ with respect to Ω and $\ker \Omega \subseteq \ker \Omega|_X$, then $\ker \Omega = \ker \Omega|_X$.*

We are now ready to prove the first theorem of this section.

Theorem 2.1. *Suppose (M, ω) is a symplectic manifold endowed with a Hamiltonian action of the Lie group G . Let $\mu \in \mathfrak{g}^*$ and K_μ its kernel group. Denote by*

$J : M \rightarrow \mathfrak{g}^*$ the associated momentum map and assume that the following hypothesis are verified:

- 1° K_μ acts properly on $J^{-1}(\mathbb{R}^+\mu)$;
- 1° J is transverse to $\mathbb{R}^+\mu$;
- 3° $\mathfrak{g} = \ker \mu + \mathfrak{g}_\mu$.

Then the ray quotient at μ

$$M_{\mathbb{R}^+\mu} := J^{-1}(\mathbb{R}^+\mu)/K_\mu$$

is a naturally symplectic orbifold, i.e. its symplectic structure $\omega_{\mathbb{R}^+\mu}$ is given by

$$\pi_{\mathbb{R}^+\mu}^* \omega_{\mathbb{R}^+\mu} = i_{\mathbb{R}^+\mu}^* \omega,$$

where

$$\pi_{\mathbb{R}^+\mu} : J^{-1}(\mathbb{R}^+\mu) \rightarrow M_{\mathbb{R}^+\mu} \quad \text{and} \quad i_{\mathbb{R}^+\mu} : J^{-1}(\mathbb{R}^+\mu) \hookrightarrow M$$

are the canonical projection and immersion respectively.

Proof. The transversality of the momentum map with respect to $\mathbb{R}^+\mu$, ensures that $J^{-1}(\mathbb{R}^+\mu)$ is a submanifold of M . Lemma 2.1 implies that the quotient $M_{\mathbb{R}^+\mu}$ is an orbifold and that $\pi_{\mathbb{R}^+\mu}$ is a surjective submersion in the category of orbifolds.

The first step is to see that the restriction of the symplectic form on $J^{-1}(\mathbb{R}^+\mu)$ is projectable on the quotient $M_{\mathbb{R}^+\mu}$. For any $\xi \in \mathfrak{k}_\mu$ and any x in M , we have that

$$T_x \pi_{\mathbb{R}^+\mu}(\xi_M(x)) = \left. \frac{d}{dt} \right|_{t=0} \pi_{\mathbb{R}^+\mu}(\exp t\xi \cdot x) = \left. \frac{d}{dt} \right|_{t=0} \pi_{\mathbb{R}^+\mu}(x) = 0.$$

Hence, $\langle \xi_{J^{-1}(\mathbb{R}^+\mu)} \mid \xi \in \mathfrak{k}_\mu \rangle \subset \ker(T\pi_{\mathbb{R}^+\mu})$. A count of dimensions shows that, in fact, the vertical distribution of $\pi_{\mathbb{R}^+\mu}$ is generated by all the infinitesimal isometries associated to the elements of \mathfrak{k}_μ . Since $\omega|_{J^{-1}(\mathbb{R}^+\mu)} = i_{\mathbb{R}^+\mu}^* \omega$ is K_μ -invariant, it follows that its Lie derivative with respect to all vector fields $\{\xi_{J^{-1}(\mathbb{R}^+\mu)} \mid \xi \in \mathfrak{k}_\mu\}$ is zero. Let $x \in J^{-1}(\mathbb{R}^+\mu)$ with $J(x) = r\mu$ and $v \in T_x(J^{-1}(\mathbb{R}^+\mu))$. Then, identifying $T_{J(x)}\mathbb{R}^+\mu$ with $\mathbb{R}\mu$, we obtain

$$\begin{aligned} \omega(i_{\mathbb{R}^+\mu}(x))(\xi_M(x), T_x i_{\mathbb{R}^+\mu} v) &= T_{i_{\mathbb{R}^+\mu}(x)} J|_{J^{-1}(\mathbb{R}^+\mu)}(v)(\xi) = \\ &= i_{\mathbb{R}^+\mu}^*(TJ|_{J^{-1}(\mathbb{R}^+\mu)})(v)(\xi) = \mu(\xi) = 0. \end{aligned}$$

It follows that $i_{\mathbb{R}^+\mu}^* \omega$ is a basic two-form which projects on $M_{\mathbb{R}^+\mu}$ to the closed form $\omega_{\mathbb{R}^+\mu} \in \Lambda^2(T^*M_\mu)$ with the property that $\pi_{\mathbb{R}^+\mu}^* \omega_{\mathbb{R}^+\mu} = i_{\mathbb{R}^+\mu}^* \omega$.

Since $\omega_{\mathbb{R}^+\mu}$ is a closed form, it remains to prove that it is also non-degenerate. For this, we will show that $T_x(K_\mu \cdot x) = \ker(i_{\mathbb{R}^+\mu}^* \omega)(x)$, for any $x \in J^{-1}(\mathbb{R}^+\mu)$. Fix $x \in J^{-1}(\mathbb{R}^+\mu)$ with $J(x) = t\mu$ and denote by $\Psi : M \rightarrow \mathfrak{k}_\mu^*$ the momentum map associated to the action of the kernel group of μ on M . Let $i^T : \mathfrak{g}^* \hookrightarrow \mathfrak{k}_\mu^*$ be the canonical inclusion. Then, $\Psi = i^T \circ J$ and $J^{-1}(\mathbb{R}^+\mu) \subset J^{-1}(\mathfrak{k}_\mu^0) = \Psi^{-1}(0)$. Notice that $J^{-1}(\mathbb{R}^+\mu) \cap G \cdot x = G_{\mathbb{R}^+\mu} \cdot x$, where $G_{\mathbb{R}^+\mu} = \{g \in G \mid \text{Ad}_g^* \mu = r\mu, r > 0\}$ is the ray isotropy group of μ . This Lie group has many interesting properties for which we refer the reader to Section 5.

For any $v \in (T_x(K_\mu \cdot x))^{\omega_x}$, $\omega_x(v, \xi_M(x)) = 0$, $\forall \xi \in \mathfrak{k}_\mu$ if and only if $T_x J(v)(\xi) = 0$, $\forall \xi \in \mathfrak{k}_\mu$. Therefore, $(T_x(K_\mu \cdot x))^{\omega_x} = T_x U$, where $U := J^{-1}(\mathfrak{k}_\mu^0) = \Psi^{-1}(0)$. We can assume U to be a submanifold of M because the transversality condition satisfied by the momentum map implies that K_μ acts locally freely at least on a neighborhood of $J^{-1}(\mathbb{R}^+\mu)$ in U , if not on the whole U .

Applying Lemma 2.2 for $(V, \Omega) := (T_x M, \omega_x)$ and $W := T_x(K_\mu \cdot x)$, we obtain that $\ker \omega_x|_{T_x U} = T_x(K_\mu \cdot x)$. We have already seen that $T_x(K_\mu \cdot x) \subset \ker i_{\mathbb{R}^+ \mu}^* \omega_x$. It follows that

$$(2.2) \quad \ker \omega_x|_{T_x U} \subset \ker \omega_x|_{T_x J^{-1}(\mathbb{R}^+ \mu)}.$$

Since $\mathfrak{g} = \ker \mu + \mathfrak{g}_\mu$, we can chose a decomposition

$$(2.3) \quad \mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m} \quad \text{with} \quad \mu|_{\mathfrak{m}} = 0.$$

Let $\mathfrak{m}_M := \{\xi_M(x) \mid \xi \in M\}$. For any $\xi \in \mathfrak{m}$ and $\eta \in \mathfrak{k}_\mu$, the equivariance of the momentum map implies that

$$T_x J(\xi_M(x))(\eta) = \xi_{\mathfrak{g}^*}(t\mu)(\eta) = -t\langle \mu, [\xi, \eta] \rangle = t\eta_{\mathfrak{g}^*}(\mu)(\xi) = 0.$$

Therefore, $\mathfrak{m}_M(x) \subset T_x U$ and $T_x J(\mathfrak{m}_M(x)) \subset T_{t\mu}(G \cdot t\mu)$. It is easy to see that

$$T_x J|_{\mathfrak{m}_M(x)}: \mathfrak{m}_M(x) \rightarrow T_{t\mu}(G \cdot t\mu)$$

is a linear isomorphism and, hence,

$$(2.4) \quad T_x J(\mathfrak{m}_M(x)) = T_{t\mu}(G \cdot t\mu).$$

Notice that equation (2.4), the third hypothesis of the theorem which can equivalently be expressed as $\{0\} = (\ker \mu)^\circ \cap (\mathfrak{g}_\mu)^\circ = \mathbb{R}\mu \cap T_{t\mu}(G \cdot t\mu)$, and the fact that $T_x J(J^{-1}(\mathbb{R}^+ \mu)) \subset \mathbb{R}\mu$ imply that

$$(2.5) \quad \mathfrak{m}_M(x) \cap T_x J^{-1}(\mathbb{R}^+ \mu) = \{0\}.$$

A simple dimension calculus shows that $\mathfrak{m}_M(x)$ and $T_x J^{-1}(\mathbb{R}^+ \mu)$ are complementary subspaces of $T_x U$. We have also seen that they are perpendicular with respect to $\omega_x|_{T_x U}$. Using relation (2.2), we can now apply Lemma 2.3 for $V := T_x M$, $W := \mathfrak{m}_M(x)$, and $X := T_x J^{-1}(\mathbb{R}^+ \mu)$. Thus, we obtain that $\ker \omega_x|_{T_x U} = T_x(K_\mu \cdot x) = \ker \omega_x|_{T_x J^{-1}(\mathbb{R}^+ \mu)}$, for any $x \in J^{-1}(\mathbb{R}^+ \mu)$. This shows that $\omega_{\mathbb{R}^+ \mu}$ is a non-degenerate form, completing thus our proof. \square

Notice that in the case $\mu = 0$ we recover the reduced symplectic space at zero. Without the hypothesis that K_μ acts properly on $J^{-1}(\mathbb{R}^+ \mu)$, the quotient $M_{\mathbb{R}^+ \mu}$ may not be Hausdorff. As the Lemma 2.1 proves, the second hypothesis of this theorem ensures that $M_{\mathbb{R}^+ \mu}$ is an orbifold. If μ is non-zero and the kernel and isotropy groups of μ coincide, then the quotient may fail to be symplectic. This always happens when the coadjoint orbit of μ is nilpotent (i.e. $\mathcal{O}_\mu = \mathcal{O}_{\mathbb{R}^+ \mu}$). As an example, consider the cotangent lift of the action of $SL(2, \mathbb{R})$ on itself by left translations. Identifying $T^*(SL(2, \mathbb{R}))$ with $SL(2, \mathbb{R}) \times sl(2, \mathbb{R})^*$ and taking as momentum value $\mu := \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \cdot \right\rangle$, one can check that $\mathcal{O}_\mu = \mathcal{O}_{\mathbb{R}^+ \mu}$. Even more,

$\ker \mu = \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & -\alpha \end{pmatrix} \mid \alpha, \gamma \in \mathbb{R} \right\} \supset \mathfrak{g}_\mu = \left\{ \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} \mid \gamma \in \mathbb{R} \right\}$. Except the last one, all the hypothesis of Theorem 2.1 are fulfilled. Since the cotangent action is free, and $\dim J^{-1}(\mathbb{R}^+ \mu) = 4$, $K_\mu = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$, the quotient $J^{-1}(\mathbb{R}^+ \mu)/K_\mu$ is 3-dimesnional, and hence not symplectic.

Corollary 2.1. *In the hypothesis of Theorem 2.1, if the dimension of M is $2n$ and the Lie group G is d -dimensional, then the dimension of the symplectic quotient is $2n - 2k - m = 2n - p - d + 2$, where $p = \dim G_\mu = k + 1$.*

In the symplectic point reduction, the reduced spaces of exact manifolds are not always exact. This is, however true, only if one performs reduction at zero momentum. Recall, for instance, that coadjoint orbits which are point reduced spaces are not necessarily exact symplectic manifolds. A counter example may be found in [23], Example (a) of Section 14.5. Surprisingly, ray quotients of exact symplectic manifolds are exact for any momentum.

Corollary 2.2. *In the hypothesis of Theorem 2.1, if $(M, \omega) = (M, -d\theta)$ with θ a K_μ -invariant one form, then the ray quotient will also be exact.*

Proof. We want to show that $i_{\mathbb{R}^+\mu}^*\theta$ is a basic form for the projection $\pi_{\mathbb{R}^+\mu} : J^{-1}(\mathbb{R}^+\mu) \rightarrow M_{\mathbb{R}^+\mu}$. The K_μ -invariance ensures that $L_{\xi_{J^{-1}(\mathbb{R}^+\mu)}} i_{\mathbb{R}^+\mu}^*\theta = 0$, for any ξ in the kernel algebra of μ . For $x \in J^{-1}(\mathbb{R}^+\mu)$, we have that

$$i_{\mathbb{R}^+\mu}^*\theta(i_\mu(x))(\xi_{J^{-1}(\mathbb{R}^+\mu)}(x)) = J(x)(\xi) = r\mu(\xi) = 0.$$

Hence, $i_{\xi_{J^{-1}(\mathbb{R}^+\mu)}}(i_{\mathbb{R}^+\mu}^*\theta) = 0$, for any $\xi \in \mathfrak{k}_\mu$, proving that $i_{\mathbb{R}^+\mu}^*\theta$ is basic. Therefore, there is a one form $\theta_{\mathbb{R}^+\mu}$ such that $i_{\mathbb{R}^+\mu}^*\theta = \pi_{\mathbb{R}^+\mu}^*\theta$. Using Theorem 2.1, we get that

$$\pi_{\mathbb{R}^+\mu}^*(-d\theta_{\mathbb{R}^+\mu}) = d(-\pi_{\mathbb{R}^+\mu}^*\theta_{\mathbb{R}^+\mu}) = -di_{\mathbb{R}^+\mu}^*\theta = i_{\mathbb{R}^+\mu}^*(-d\theta) = i_{\mathbb{R}^+\mu}^*\omega = \pi_{\mathbb{R}^+\mu}^*\omega_{\mathbb{R}^+\mu}.$$

Since $\pi_{\mathbb{R}^+\mu}^*$ is injective we obtain that $\omega_{\mathbb{R}^+\mu} = -d\theta_{\mathbb{R}^+\mu}$. \square

A large class of examples can be obtained in the case when (M, ω) is the cotangent bundle of a manifold Q endowed with the canonical symplectic form $\omega_0 = -d\theta_0$. We treat this case in Section 3, Corollary 3.1.

We will now extend this reduction procedure to the metric context, i.e. for Kähler manifolds.

Theorem 2.2. *Let (M, g, ω) be a Kähler manifold and G a Lie group acting on M by Hamiltonian symplectomorphisms. If $J : M \rightarrow \mathfrak{g}^*$ is the momentum map associated to the action of G and μ an element of \mathfrak{g}^* , assume that:*

- 1° $\text{Ker } \mu + \mathfrak{g}_\mu = \mathfrak{g}$;
- 2° the action of K_μ on $J^{-1}(\mathbb{R}^+\mu)$ is proper and by isometries;
- 3° J is transverse to $\mathbb{R}^+\mu$.

Then the ray quotient at μ

$$M_{\mathbb{R}^+\mu} := J^{-1}(\mathbb{R}^+\mu)/K_\mu$$

is a Kähler orbifold with respect to the projection of the metric g .

Proof. From Theorem 2.1, we already know that $(M_{\mathbb{R}^+\mu}, \omega_{\mathbb{R}^+\mu})$ is a symplectic orbifold. It remains to show that the symplectic structure is also a Kähler one with corresponding metric given by the projection of g . The second hypothesis of the theorem ensures that $(J^{-1}(\mathbb{R}^+\mu), i_{\mathbb{R}^+\mu}^*g)$ is an isometric Riemannian submanifold of M .

Again, we will use a decomposition $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m}$, where $\mu|_{\mathfrak{m}} = 0$. Let $\Psi : M \rightarrow \mathfrak{k}_\mu^*$ be the momentum map associated to the action of K_μ on M and $\mathfrak{m}_M := \{\xi_M(x) \mid \xi \in M\}$. In the proof above we have already seen that

$$(2.6) \quad T_x J^{-1}(\mathbb{R}^+\mu) \oplus \mathfrak{m}_M(x) = T_x \Psi^{-1}(0),$$

for any $x \in J^{-1}(\mathbb{R}^+\mu)$. Let $\{\xi_1, \dots, \xi_k\}$ and $\{\eta_1, \dots, \eta_m\}$ be basis in \mathfrak{k}_μ and \mathfrak{m} respectively, with $m = \dim \mathfrak{m}$ and $k = \dim \mathfrak{k}_\mu$. Without loss of generality, we can assume that the infinitesimal isometries $\{\xi_{iM}\}_{i=1,k}$ and $\{\eta_{jM}\}_{j=1,m}$ are g -orthogonal.

Thus, $\{J\xi_{iM}, J\eta_{jM}\}_{i,j}$ are linearly independent in each point of $J^{-1}(\mathbb{R}^+\mu)$. Even more, $\{J\xi_{iM}, J\eta_{jM}\}_{i,j}$ belong to the normal fiber bundle of $J^{-1}(\mathbb{R}^+\mu)$ since

$$g(J\eta_{jM}, V) = g(J\xi_{iM}, V) = \omega(\xi_{iM}, V) = -TJ(V)(\xi) = -r\mu(\xi_i) = -r\mu(\eta_j) = 0$$

for any V vector field on $J^{-1}(\mathbb{R}^+\mu)$. The next step is to show that $\{J\xi_{iM}\}_{i=1,k}$ is a basis in the normal bundle of $T\Psi^{-1}(0)$. Notice that $\{\xi_{iM} \mid_{J^{-1}(\mathbb{R}^+\mu)}\}_{i=1,k}$ are tangent to $J^{-1}(\mathbb{R}^+\mu)$ and

$$g(J\xi_{iM}, V) = \omega(\xi_{iM}, V) = T\Psi(V)(\xi_i) = Ti^T(TJ(V))(\xi_i) = 0,$$

for any V differentiable section of $T\Psi^{-1}(0)$. Here, we have used that $\Psi = i_T^* \circ J$, where $i_T^* : \mathfrak{g}^* \rightarrow \mathfrak{k}_\mu^*$ is the canonical projection. Therefore, $\{J\xi_{iM}\}_{i=1,k}$ are vector fields normal to TU , where $U = J^{-1}(k_\mu^\circ) = \Psi^{-1}(0)$. As $\dim TU = \dim M - \dim \mathfrak{k}_\mu$, these vector fields form a basis of the normal fiber bundle to TU . Equation (2.6) implies that $\{J\xi_{iM}, J\eta_{jM}\}_{i,j}$ form a basis of the normal bundle to $J^{-1}(\mathbb{R}^+\mu)$. Since the action of K_μ on $J^{-1}(\mathbb{R}^+\mu)$ is isometric, $i_{\mathbb{R}^+\mu}^* g$ projects on $M_{\mathbb{R}^+\mu}$ in $\mathfrak{g}_{\mathbb{R}^+\mu}$ and the projection $\pi_{\mathbb{R}^+\mu}$ becomes thus a Riemannian submersion. Obviously, the vertical distribution of this Riemannian submersion is given by $\{\xi_{iM}\}_{i=1,k}$. Then, $T_x J^{-1}(\mathbb{R}^+\mu) = \{\xi_{iM}\}(x) \oplus \mathcal{H}_x$, where \mathcal{H}_x is the horizontal distribution at x associated to the Riemannian submersion π_μ . To see that $(\omega_{\mathbb{R}^+\mu}, \mathfrak{g}_{\mathbb{R}^+\mu})$ is an almost Kähler structure, we need to check that

$$\omega_{\mathbb{R}^+\mu}([x])(T_x \pi_\mu v, T_x \pi_\mu w) = \mathfrak{g}_{\mathbb{R}^+\mu}([x])(\mathcal{C}_{\mathbb{R}^+\mu} T_x \pi_\mu v, T_x \pi_\mu w),$$

for any $[x] = \pi_\mu(x) \in J^{-1}(\mathbb{R}^+\mu)$ and $v, w \in \mathcal{H}_x$. Here, $\mathcal{C}_{\mathbb{R}^+\mu}$ denotes the projection of the complex structure \mathcal{C} of ω . Since $T_x \pi_\mu$ is an isomorphism from the horizontal space at x onto $T_{[x]} M_{\mathbb{R}^+\mu}$ which identifies $(\omega_{\mathbb{R}^+\mu}, \mathfrak{g}_{\mathbb{R}^+\mu})([x])$ with $(i_{\mathbb{R}^+\mu}^* \omega, i_{\mathbb{R}^+\mu}^* g) \mid_{\mathcal{H}_x}$ suffices to show that the horizontal distribution is \mathcal{C} -invariant. Let $v \in \mathcal{H}_x$. Then $\omega(\mathcal{C}v, \xi_{iM}) = g(v, \xi_{iM}) = 0$, for any $\xi_i \in \mathfrak{k}_\mu$. Also $g(\mathcal{C}v, \mathcal{C}\xi_{iM}) = g(v, \xi_{iM}) = 0$ and $g(\mathcal{C}v, \mathcal{C}\eta_{jM}) = g(v, \eta_{jM}) = 0$, for all $i = 1, k$ and $j = 1, m$. It follows that $\mathcal{C}v$ is also a horizontal vector. To show that $\mathcal{C}_{\mathbb{R}^+\mu}$ is integrable we will evaluate the Nijenhuis tensor $N_{\mathbb{R}^+\mu}$. Thus,

$$\begin{aligned} N_{\mathbb{R}^+\mu}(T_x \pi_\mu(v), T_x \pi_\mu(w)) &= [T_x \pi_\mu(v), T_x \pi_\mu(w)] - [\mathcal{C}_{\mathbb{R}^+\mu} T_x \pi_\mu(v), \mathcal{C}_{\mathbb{R}^+\mu} T_x \pi_\mu(w)] \\ &\quad + \mathcal{C}_{\mathbb{R}^+\mu}([\mathcal{C}_{\mathbb{R}^+\mu} T_x \pi_\mu(v), T_x \pi_\mu(w)]) + \mathcal{C}_{\mathbb{R}^+\mu}([T_x \pi_\mu(v), \mathcal{C}_{\mathbb{R}^+\mu} T_x \pi_\mu(w)]) \\ &= T_x \pi_\mu([v, w]) - T_x \pi_\mu([\mathcal{C}v, \mathcal{C}w]) + \mathcal{C}_{\mathbb{R}^+\mu}(T_x \pi_\mu([\mathcal{C}v, w])) + \mathcal{C}_{\mathbb{R}^+\mu}(T_x \pi_\mu([v, \mathcal{C}w])) \\ &= T_x \pi_\mu([v, w] - [\mathcal{C}v, \mathcal{C}w]) + T_x \pi_\mu(\mathcal{C}([\mathcal{C}v, w])) + T_x \pi_\mu(\mathcal{C}([v, \mathcal{C}w])) \\ &= T_x \pi_\mu(N(v, w)) = 0, \end{aligned}$$

where N is the Nijenhuis tensor of (ω, g) . Thus, $\mathcal{C}_{\mathbb{R}^+\mu}$ is integrable and $(M_{\mathbb{R}^+\mu}, \omega_{\mathbb{R}^+\mu}, \mathfrak{g}_{\mathbb{R}^+\mu})$ a Kähler manifold. \square

Remark 2.1. Unfortunately, non zero Kähler regular point reduction is not canonical. As it is very well explained in [6] (see Exercise 3), the complex structure may not leave invariant the horizontal distribution of the Riemannian submersion given by the quotient projection $(\pi_\mu : M \rightarrow M_\mu)$. Therefore it is not projectable on M_μ . The solution proposed in the literature, is based on the Shifting Theorem (see Theorem 6.5.2 in [28]). More precisely, one endows the coadjoint orbit of μ , \mathcal{O}_μ with a unique up to homotheties Kähler-Einstein metric of positive Ricci curvature. For the construction of this metric, see [21], Chapter 8 in [2], and [19]. This uniqueness modulo homotheties is guaranteed by the choice of an Ad^* -invariant scalar product

on \mathfrak{g}^* . Then, one performs the zero reduction of the Kähler difference of the base manifold M and \mathcal{O}_μ . Unfortunately, this construction is correct only in the case of totally isotropic momentum (i.e. $G_\mu = G$). Otherwise, using the unique Kähler-Einstein form on the coadjoint orbit, instead of the Kostant-Kirillov-Souriau form makes impossible the use of the Shifting Theorem since the momentum map of the orbit will no longer be the inclusion. Even so, one could take by definition the reduced space at μ momentum to be the zero reduced space of the symplectic difference of M and \mathcal{O}_μ . But this reduced space is not canonical, in the sense that the pull-back through the quotient projection of the reduced Kähler structure is no longer the initial one. On the other hand, the ray Kähler reduction always exists and is canonical.

3. CONE AND BOOTHBY-WANG COMPATIBILITIES

Traditionally, Sasakian manifolds were defined via contact structures by adding a Riemannian metric with certain compatibility conditions.

Definition 3.1. A Sasakian structure on an exact contact manifold (S, η, \mathfrak{R}) is a Riemannian metric g on S such that there is a $(1, 1)$ -tensor field Φ which verifies the following identities

$$\Phi^2 = -Id + \eta \otimes \mathfrak{R} \quad \eta(X) = g(X, \mathfrak{R}) \quad d\eta(X, Y) = g(X, \Phi Y),$$

for any vector fields X, Y .

A good reference for this point of view is the book of D. E. Blair, [3].

There are other equivalent definitions of a Sasakian manifold and in the following proposition we present four of them. The first one is most in the spirit of the original definition of Sasaki (see [31]). The most geometric approach is highlighted in the second definition. It only uses the holonomy reduction of the associated cone metric and it was introduced by C. P. Boyer and K. Galicki in [5].

Proposition 3.1. Let (S, g) be a Riemannian manifold of dimension m , ∇ the associated Levi-Civita connection, and R the Riemannian curvature tensor of ∇ . Then, the following statements are equivalent:

- there exists a unitary Killing vector field \mathfrak{R} on S so that the tensor field Φ of type $(1, 1)$, defined by $\Phi(X) = \nabla_X \mathfrak{R}$, satisfies the condition

$$(\nabla_X \Phi)(Y) = g(\mathfrak{R}, Y)X - g(X, Y)\mathfrak{R},$$

for any pair of vector fields X and Y on S ;

- the holonomy group of the cone metric on S , $(\mathcal{C}(S), \mathcal{C}(g)) := (S \times \mathbb{R}^+, r^2 g + dr^2)$ reduces to a subgroup of $U(\frac{m+1}{2})$. In particular, $m = 2n + 1$, for a $n \geq 1$ and $(\mathcal{C}(S), \mathcal{C}(g))$ is Kähler;
- there exists a unitary Killing vector field \mathfrak{R} on S so that the Riemannian curvature satisfies the condition

$$R(X, \mathfrak{R})Y = g(\mathfrak{R}, Y)X - g(X, Y)\mathfrak{R},$$

for any pair of vector fields X and Y on S ;

- there exists a unitary Killing vector field \mathfrak{R} on S so that the sectional curvature of every section containing \mathfrak{R} equals one;
- (S, g) is a Sasakian manifold.

For the proof, see [5].

Example: Sasakian spheres. One of the simplest compact examples of Sasakian manifolds is the standard sphere $S^{2n+1} \subset \mathbb{C}^n$ with the metric induced by the flat one on \mathbb{C}^n . The characteristic Killing vector field (i.e. the associated Reeb vector field) is given by $\Re(p) = -i\vec{p}$, i being the imaginary unit. The contact form is given by $\eta := \frac{1}{2}(dz - \sum^n y^j dx_j)$, if $(x_j, y^j, z)_{j=1,n}$ are the canonical coordinates on the base space.

Recall that if (M, η) is a $2n+1$ -dimensional exact contact manifold, its symplectic cone is given by $\mathcal{C}(M) := (M \times \mathbb{R}^+, dr^2 \wedge \eta + r^2 d\eta)$ and M can be embedded in the cone as $M \times \{1\}$. The cone of a Sasakian (S, g) manifold admits a canonical Kähler structure given by $\mathcal{C}(g) := r^2 g + dr^2$. If a Lie group G acts by contact isometries on S , then this action can be lifted to the Kähler cone as $g \cdot (x, r) := (g \cdot x, r)$, for any $g \in G$ and $(x, r) \in \mathcal{C}(S)$. This action commutes with the translations on the \mathbb{R}^+ component and, in the Sasakian case, it is by holomorphic isometries. In the Sasakian case, we can also define a complex structure given as follows:

$$\mathcal{C}Y := \varphi Y - \eta(Y)R, \mathcal{C}R := \xi,$$

where $R = r\partial_r$ is the vector field generated by the 1-group of transformations $\rho_t : (x, r) \rightarrow (x, tr)$ and $\varphi := \nabla \xi$, with ∇ the Levi-Civita connection associated to g . It is easy to see that (S, η, g) is Einstein if and only if the cone metric $\mathcal{C}(g)$ is Ricci flat, i. e., $(\mathcal{C}(S), \mathcal{C}(g))$ is Calabi-Yau (i. e. Kähler Ricci-flat).

Let $\Phi : S \rightarrow \mathfrak{g}^*$ be the contact momentum map associated to the G -action on S . The lifted action on the cone is Hamiltonian and a corresponding equivariant symplectic momentum map is given by

$$\Phi_s : \mathcal{C}(S) \rightarrow \mathfrak{g}^*, \Phi_s(x, r) := e^s J(x), \text{ for any } (x, r) \in \mathcal{C}(S).$$

Having established the above notations, we are ready to prove that reduction and the cone construction are commuting operations.

Lemma 3.1. *Let (S, η, g, ξ) be a Sasakian manifold and $(\mathcal{C}(S), \mathcal{C}(g), J)$ its Kähler cone. Suppose a Lie group G acts on S by strong contactomorphisms and commuting with the action of the 1-parameter group generated by the field R . Let μ be an element of the dual of the Lie algebra of G . Then the Kähler cone of the reduced contact space at μ is the reduced space at μ for the lifted action on $\mathcal{C}(S)$.*

Proof. Let K_μ be the kernel group of μ , $(S_{\mathbb{R}^+ \mu}, \eta_{\mathbb{R}^+ \mu}, g_{\mathbb{R}^+ \mu})$ the corresponding contact reduced space, and $\mathcal{C}(S_{\mathbb{R}^+ \mu})$ the reduced space for the lift of the action on the cone. Since the K_μ -action commutes with homotheties on the \mathbb{R}^+ component, there is a natural diffeomorphism between $\mathcal{C}(S_{\mathbb{R}^+ \mu})$ and $\mathcal{C}(S)_{\mathbb{R}^+ \mu}$:

$$\Psi : \mathcal{C}(S)_{\mathbb{R}^+ \mu} \rightarrow \mathcal{C}(S_{\mathbb{R}^+ \mu}), \Psi([x, r]) := ([x], r), \forall [x, r] \in \mathcal{C}(S)_{\mathbb{R}^+ \mu}.$$

Using the commutativity of the diagram of Figure 1, it is easy to see that Ψ is also a symplectomorphic isometry. Namely,

$$(\Psi \circ \pi_{1\mathbb{R}^+ \mu})^*(\eta_{\mathbb{R}^+ \mu} \wedge dr^2 + r^2 d\eta_{\mathbb{R}^+ \mu}) = i_{1\mathbb{R}^+ \mu}^*(\eta \wedge dr^2 + r^2 d\eta),$$

and

$$\Psi^*(\mathcal{C}(g_{\mathbb{R}^+ \mu})) = \mathcal{C}(g)_{\mathbb{R}^+ \mu},$$

where $i_{1\mathbb{R}^+ \mu} : \Phi_s^{-1}(\mathbb{R}^+ \mu) \rightarrow \mathcal{C}(S)$, $\pi_{1\mathbb{R}^+ \mu} : \Phi_s^{-1}(\mathbb{R}^+ \mu) \rightarrow \mathcal{C}(S)_{\mathbb{R}^+ \mu}$, and $\pi_{\mathbb{R}^+ \mu} : \Phi^{-1}(\mathbb{R}^+ \mu) \rightarrow (S)_{\mathbb{R}^+ \mu}$ are the canonical inclusion and K_μ -projections, respectively. \square

$$\begin{array}{ccccc}
\Phi^{-1}(\mathbb{R}^+\mu) \times \mathbb{R}^+ & \xrightarrow{\pi_{1\mathbb{R}^+\mu}} & \mathcal{C}(S)_{\mathbb{R}^+\mu} & \xrightarrow{\Psi} & S_{\mathbb{R}^+\mu} \times \mathbb{R}^+ \\
\downarrow \pi_{\mathbb{R}^+\mu} \times \text{id}_{\mathbb{R}^+} & & \nearrow \text{id}_{S_{\mathbb{R}^+\mu} \times \mathbb{R}^+} & & \\
S_{\mathbb{R}^+\mu} \times \mathbb{R}^+ & \xrightarrow{\simeq} & & &
\end{array}$$

FIGURE 1. Commutative diagram used in the proof of Lemma 3.1

Corollary 3.1. *Let Q be a differentiable manifold of real dimension n , G a finite dimensional Lie group acting smoothly on Q . Denote by μ an element of the dual Lie algebra \mathfrak{g}^* and by K_μ its kernel group. Assume that K_μ acts freely and properly on $J^{-1}(\mathbb{R}^+\mu)$, with $J : T^*Q \rightarrow \mathfrak{g}^*$ the canonical momentum map associated to the G -action. Then the ray reduced space $(T^*(Q))_{\mathbb{R}^+\mu}$ is embedded by a map preserving the symplectic structures onto a subbundle of $T^*(Q/K_\mu)$.*

Proof. Note that the symplectic cone of the cosphere bundle of Q is exactly $T^*Q \setminus \{o_{T^*Q}\}$. Applying Theorems 3.1 and 3.2 in [9], and the above lemma the conclusion of the Corollary follows. \square

Recall that a celebrated theorem of Boothby and Wang (see Section 3.3 in [3]) states that if the contact manifold (M, η) is also compact and regular, then it admits a contact form whose Reeb vector field generates a free, effective S^1 -action on it. A contact structure is regular if it admits a regular Reeb vector field \mathfrak{R} , i.e. any point in M has a cubical neighborhood such that all the integral curves of \mathfrak{R} pass at most once through this neighborhood. Even more, M is the bundle space of a principal circle bundle $\pi : M \rightarrow N$ over a symplectic manifold of dimension $2n$ with symplectic form ω determining an integer cocycle. In this case, η is a connection form on the bundle $\pi : M \rightarrow N$ with curvature form $d\eta = \pi^*\omega$. N is actually the space of leaves of the characteristic foliation on M (i.e. the 1-dimensional foliation defined by the Reeb vector field of η). If $M = S$ is a Sasakian manifold, then N becomes a Hodge manifold and the fibers of π are totally geodesic. This case was treated by Y. Hatakeyama in [15]. Even more, in [4], Theorem 2.4 it was proved that S is Sasakian-Einstein if and only if N is Kähler-Einstein with scalar curvature $4n(n+1)$ and that all the above still holds in the category of orbifolds if S is quasi-regular, i.e. all the leaves of the characteristic foliation are compact.

Proposition 3.2. *Let $\pi : (S, \mathfrak{g}) \rightarrow (N, \mathfrak{h})$ be the Boothby-Wang fibration associated to the quasi-regular, compact, Sasakian manifold S . Suppose a connected Lie group G acts by strong contactomorphisms on (S, \mathfrak{g}) with momentum map $J_S : S \rightarrow \mathfrak{g}^*$. Let μ be an element of \mathfrak{g}^* , with kernel group K_μ . Assume that the action of K_μ on $J^{-1}(\mathbb{R}^+\mu)$ is proper and by isometries and that $\ker \mu + \mathfrak{g}_\mu = \mathfrak{g}$. Then, the reduced space of N at μ is well defined and there is a canonical Boothby-Wang fibration of the reduced spaces:*

$$\tilde{\pi} : S_{\mathbb{R}^+\mu} \rightarrow N_{\mathbb{R}^+\mu}.$$

Proof. Denote by η the contact form of the Boothby-Wang fibration and by \mathfrak{R} its Reeb vector field. Since $[\mathfrak{R}, \xi_S] = 0$ for any $\xi \in \mathfrak{g}$ and G is connected, the action

generated by the Reeb vector field commutes with the action of G . Hence there is a well defined action of G on N . Even more, this action is by symplectomorphisms. If $J_S : S \rightarrow \mathfrak{g}^*$ is the equivariant momentum map associated to the G -action on S , the induced application

$$J_N : N \rightarrow \mathfrak{g}^*, J_N(\pi(x)) := J_S(x),$$

is well defined for any $x \in S$. Indeed, if Φ_{\Re}^t is the flow of the Reeb vector field, we have

$$\begin{aligned} J_S(\Phi_{\Re}^t(x))(\xi) &= \eta(\Phi_{\Re}^t(x))(\xi_S(\Phi_{\Re}^t(x))) = ((\Phi_{\Re}^t)^*\eta)(x)(\xi_S(x)) = \eta(x)(\xi_S(x)) \\ &= J_S(x)(\xi), \end{aligned}$$

for any $\xi \in \mathfrak{g}$ and any $x \in S$. This proves that J_N is well defined. Using the fact that $\pi^*\omega = d\eta$, it is easy to see that J_N is an equivariant momentum map associated to the G -action on N . We also have that $\pi(J_S^{-1}(\mathbb{R}^+\mu)) = J_N^{-1}(\mathbb{R}^+\mu)$ and obviously the action of K_μ on $J_N^{-1}(\mathbb{R}^+\mu)$ is proper and by isometries. Therefore, the quotient space $N_{\mathbb{R}^+\mu}$ is a well defined symplectic orbifold and the induced projection $\tilde{\pi} : S_{\mathbb{R}^+\mu} \rightarrow N_{\mathbb{R}^+\mu}$ becomes a Boothby-Wang fibration. \square

4. CONFORMAL HAMILTONIAN VECTOR FIELDS

In this section we will study the dynamical behavior of conformal Hamiltonian systems. This class of systems comprises mechanical, non autonomous systems with friction or Rayleigh dissipation. The definition of conformal Hamiltonian vector fields appeared for the first time in the work of McLachlan and Perlmutter, see [22]. In this section we will see that in the presence of symmetries the solutions of conformal Hamiltonian systems preserve the ray pre-images of the momentum map, but not the point pre-images used in the construction of the Marsden-Weinstein quotient. Therefore, the right tool for the study of symmetries of these systems is the ray reduction and not the point one. We will also enlarge the class of conformal Hamiltonian systems previously defined and we will complete their Lie-Poisson reduction with the general ray reduction.

Recall that the energy of autonomous Hamiltonian systems is conserved. If they are endowed with an appropriate symmetry group G , then they also obey an other conservation law. Namely, if $H \in \mathcal{C}^\infty(M)$ is the G -invariant Hamiltonian, $J : M \rightarrow \mathfrak{g}^*$ an associated equivariant momentum map, the pre-images $\{J^{-1}(\mu) | \mu \in \mathfrak{g}^*\}$ are invariant submanifolds of the Hamiltonian vector field. In symplectic geometry this conservation property is known as the *Noether theorem* and it states that if $t \rightarrow c(t)$ is a solution of the Hamiltonian system starting at the point x_0 with momentum $J(x_0) = \mu$, then at any time t the solution will have the same momentum μ . In other words, the Hamiltonian flow leaves the connected components of $J^{-1}(\mu)$ invariant and commutes with the group action. Hence, it projects on M_μ onto another Hamiltonian flow corresponding to the smooth function $H_\mu \in \mathcal{C}^\infty(M_\mu)$ defined by $H_\mu \circ \pi_\mu = H \circ i_\mu$. The triple $(M_\mu, \omega_\mu, X_{H_\mu})$ is called the *reduced Hamiltonian system*. Of course, in this setup appropriate symmetries refer to a proper, free action which ensures the smoothness of the quotient M_μ . This is a classical result of J. Marsden and A. Weinstein. For the proof and physical examples, see [24] and [25].

However, in physics there are a lot of simple mechanical systems whose energy is not conserved, but dissipated. One class of such systems is the class of conformal

Hamiltonians. In [22] and in the following paragraph we will briefly recall the definition and some of their properties. After, we will show how to extend the class of conformal Hamiltonian systems.

In this section $(M, \omega = -d\theta)$ will be an exact symplectic manifold. The vector field X_H^k on M is *conformal with real parameter k* if $i_{X_H^k} \omega = dH - k\theta$ for a smooth Hamiltonian H . This condition is equivalent to $L_{X_H^k} = -k\omega$. Note that the hypothesis of exactness of the symplectic form does not restrain the generality since a symplectic manifold admits a vector field X_H^k with $L_{X_H^k} = -k\omega$ if and only if it is exact. If, in addition, $H^1(M) = 0$, then all the conformal vector fields on M are given by

$$\{X_H + kZ \mid H \in C^\infty(M)\},$$

where Z is the Liouville vector field defined by $i_Z \omega = -\theta$. For the proof, see Proposition 1 in [22]. It was noticed by the authors of this article that, in the case of Lie group symmetries, the conformal Hamiltonian vector fields have a special behaviour with respect to the associated momentum map. Namely,

Proposition 4.1. *Let G be a Lie group which acts on $(M, \omega = -d\theta)$ leaving the 1-form θ invariant and H a smooth, G -invariant function on M . Denote by $J : M \rightarrow \mathfrak{g}^*$ the associated G -equivariant momentum map. Then, X_H^k is a G -invariant vector field for any real k and its flows preserves the ray pre-images of the associated momentum map as follows:*

$$J(x(t)) = e^{-kt} J(x(0)),$$

for any integral curve x of X_H^k and any time t .

In other words, the motion is constrained to a ray of momentum values entirely determined by the initial momentum. Hence, the ray pre-images of the momentum map are invariant submanifolds for the conformal Hamiltonian vector fields. In the hypothesis of Proposition 4.1, with M the cotangent bundle of a Lie group G , the authors have performed the conformal Lie Poisson reduction and reconstruction of solutions for conformal Hamiltonian vector fields. However, they could not exploit the ray momentum conservation, nor perform a reduction which uses not only the group invariance, but also the ray-momentum one. Proposition 4.1 and Theorem 2.1, immediately suggest that the appropriate method of reduction for conformal Hamiltonian vector fields is the ray reduction constructed in Section 2.

But before passing to details, we want to show how to generalize the definition of conformal Hamiltonian vector fields in order to include in this study more physical systems. Let us first recall the example of Rayleigh systems. On the canonical symplectic manifold $(\mathbb{R}^{2n}, q, p, \omega = dq \wedge dp)$ they are defined by

$$(4.1) \quad \begin{cases} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} - R(q) \frac{\partial H}{\partial p}, \end{cases}$$

where $H = T + V(q)$, $T = \frac{1}{2}p^T M(q)p$, M positive definite. If R is positive, they dissipate energy since $dH = -R(q) \langle \frac{\partial H}{\partial p}, \frac{\partial H}{\partial p} \rangle$. Of course the system (4.1) is conformal Hamiltonian with parameter k if and only if $R(q) = kM(q)^{-1}$. However, if the real parameter k is replaced by the real function $f(q, p)$, then the vector field defining (4.1) is characterized by the equality $i_X \omega = dH - f\theta$.

These examples suggest the following enlarged definition of a conformal Hamiltonian vector field on an exact symplectic manifold.

Definition 4.1. The vector field X_H^f on the symplectic manifold $(M, \omega = -d\theta)$ is *conformal Hamiltonian* with conformal parameter the smooth function f and smooth Hamiltonian H if $i_{X_H^f} \omega = dH - f\theta$.

Remark 4.1. Observe that if $H^1(M) = \{0\}$, X_H^f is conformal Hamiltonian if and only if $L_{X_H^f} \omega = -d(f\theta)$.

Remark 4.2. The conformal Hamiltonian $X_H^f = X_H + Z_f$ is the sum of the Hamiltonian vector field determined by H and the vector field uniquely determined by the relation $i_Z \omega = -f\theta$. In local coordinates (q, p) , Z is given by $fp \frac{\partial}{\partial p}$.

The next proposition shows that this enlarged class of conformal Hamiltonians behaves well in the presence of symmetries.

Proposition 4.2. Let G be a Lie group which acts on $(M, \omega = -d\theta)$ leaving the 1-form θ invariant, H and f smooth, G -invariant functions on M . Denote by $J : M \rightarrow \mathfrak{g}^*$ the associated G -equivariant momentum map. Then, X_H^f is a G -invariant vector field and its flow preserves the ray pre-images of the associated momentum map as follows:

$$J(x(t)) = e^{\int_0^t -f(x(s)) ds} J(x(0)),$$

for any integral curve x of X_H^f and any time t .

Proof. Denote by ϕ the action of G on M . Then, for any $g \in G$ we have

$$(4.2) \quad \phi_g^*(i_{X_H^f} \omega) = \phi_g^*(dH - f\theta) = dH - f\theta = i_{X_H^f} \omega,$$

since f and H are G -invariant. On the other hand,

$$(4.3) \quad \phi_g^*(i_{X_H^f} \omega) = i_{\phi_g^* X_H^f} \phi_g^* \omega = i_{\phi_g^* X_H^f} \omega.$$

Since ω is non-degenerate, (4.2) and (4.3) imply that X_H^f is G -invariant.

First recall that any exact symplectic manifold admits an equivariant momentum map given by $J : (M, \omega = d\theta) \rightarrow \mathfrak{g}^*$, $\langle J(x), \xi \rangle := \theta(\xi_M)(x)$, for any $x \in M$ and $\xi \in \mathfrak{g}$. Now, let $x(t)$ be an integral curve of X_H^f . Then,

$$\begin{aligned} \frac{d}{dt} \langle J(x(t)), \xi \rangle &= T J^\xi(X_H^f(x(t))) = \omega(x(t))(X_H^f(x(t)), \xi_M(x(t))) = \\ &= dH(\xi_M(x(t)) - f(x(t))\theta(\xi_M(x(t)))) = -f(x(t))J^\xi(x(t)). \end{aligned}$$

Hence, $J^\xi(x(t)) = e^{\int_0^t -f(x(s)) ds} J^\xi(x(0))$ for any $\xi \in \mathfrak{g}$ and any time t . \square

Remark 4.3. Note that if θ , f , and H are K_μ -invariant, with K_μ the kernel group associated to $\mu \in \mathfrak{g}^*$, then the corresponding conformal Hamiltonian is also K_μ -invariant. Even more, if the K_μ -action is proper and free X_H^f projects onto a conformal Hamiltonian with parameter function and Hamiltonian canonically induced by f and H .

Definition 4.2. If in the hypothesis of the above remark one replaces K_μ with G , the point $x \in M$ is called a *relative equilibrium* (or *relative periodic*) point of X_H^f if it descends through the projection $M \mapsto M/G$ onto an equilibrium (or periodic) point of the reduced conformal Hamiltonian.

Proposition 4.2 suggests that the ray reduction is a natural tool for the study of conformal Hamiltonian systems. Indeed,

Proposition 4.3. *Consider $(M, \omega = -d\theta)$ an exact symplectic manifold endowed with the smooth action of a Lie group G . Choose an element μ in \mathfrak{g}^* with kernel group K_μ . Denote by $J : M \rightarrow \mathfrak{g}^*$ the associated equivariant momentum map defined by $J(x)(\xi) := i_{\xi_M}\theta$, for any $x \in M$ and $\xi \in \mathfrak{g}$ with infinitesimal isometry ξ_M . Suppose that all the hypothesis of Theorem 2.1 are fulfilled and X_H^f is a conformal Hamiltonian vector field with H and f K_μ -invariant functions. Then,*

- the flow of X_H^f induces a flow on the ray reduced space $M_{\mathbb{R}^+\mu}$ defined by

$$\pi_{\mathbb{R}^+\mu} \circ \Phi_t \circ i_{\mathbb{R}^+\mu} = \Phi_t^{\mathbb{R}^+\mu} \circ \pi_{\mathbb{R}^+\mu}.$$

- the vector field generated by the flow $\Phi_t^{\mathbb{R}^+\mu}$ is conformal Hamiltonian $(X_H^f)_{\mathbb{R}^+\mu}$ with

$$f_{\mathbb{R}^+\mu} \circ \pi_{\mathbb{R}^+\mu} = f \circ i_{\mathbb{R}^+\mu}, H_{\mathbb{R}^+\mu} \circ \pi_{\mathbb{R}^+\mu} = H \circ i_{\mathbb{R}^+\mu}.$$

The vector fields X_H^f and $(X_H^f)_{\mathbb{R}^+\mu}$ are $\pi_{\mathbb{R}^+\mu}$ -related.

- a point $x \in M$ with momentum μ is a relative equilibrium of X_H^f if and only if there is an element ξ of the ray isotropy algebra $\mathfrak{g}_{\mathbb{R}^+\mu}$ such that $X_H^f(x) = \xi_M(x)$ or, equivalently, $\Phi_t(x) = \exp t\xi \cdot x$, for any time t . The relative equilibria of X_H^f with momentum μ coincide via the $\pi_{\mathbb{R}^+\mu}$ -projection with the equilibria of $(X_H^f)_{\mathbb{R}^+\mu}$, or, equivalently, with the points $x \in M$ with momentum μ for which there is a $\xi \in \mathfrak{g}_{\mathbb{R}^+\mu}$ such that

$$(4.4) \quad d(J^\xi - H)(x) = f(x)\theta(x).$$

- a point $x \in M$ with momentum μ is a relative periodic point of X_H^f if and only if there is an element g of the kernel group K_μ and a positive constant τ such that $\Phi_{t+\tau}(x) = g\Phi_t(x)$ at any time t .

Remark 4.4. Note that, in local symplectic coordinates (q, p) , condition (4.4) is equivalent to

$$(4.5) \quad \begin{cases} pf &= \frac{\partial(J^\xi - H)}{\partial q} \\ 0 &= -\frac{\partial(J^\xi - H)}{\partial p}. \end{cases}$$

Proof. The first two points of the theorem are a direct consequence of Proposition 4.2. For the rest, suffice it to use the definition of a conformal Hamiltonian vector field, the relation $\omega(\xi_M, \cdot) = dJ^\xi(\cdot)$, and Proposition 4.2. \square

Example 4.1. The reduction of a Rayleigh system on $T^*(\mathbb{R}^{2*} \times \mathbb{R}^{2*})$.

On $(T^*(\mathbb{R}^{2*} \times \mathbb{R}^{2*}), (q, p)) \simeq ((\mathbb{R}^{2*} \times \mathbb{R}^{2*}) \times \mathbb{R}^4, (q_1, q_2, p_1, p_2))$ consider the Rayleigh system given by $H(q, p) = \frac{1}{2}(\|q\|^2 + \|p\|^2)$ and $f(q, p) = \|q_1\|^2 + \|p_1\|^2$. Consider the cotangent lift of the rotation action of $S^1 \times S^1$ on $\mathbb{R}^{2*} \times \mathbb{R}^{2*}$. The reason for restricting \mathbb{R}^4 to $(\mathbb{R}^{2*} \times \mathbb{R}^{2*})$ is to have free symmetries. The action is also proper and H and f are $S^1 \times S^1$ -invariant. Let $\mu := \langle (0, 1), \cdot \rangle$ be an element of $(\mathbb{R} \times \mathbb{R})^*$, the dual of the Lie algebra of $S^1 \times S^1$. Then $K_\mu = \{e\} \times S^1$ and $\mathfrak{k}_\mu = \{0\} \times \mathbb{R}$. The momentum map associated to the $S^1 \times S^1$ -action is given by

$$J : \mathbb{R}^{2*} \times \mathbb{R}^{2*} \times \mathbb{R}^4 \rightarrow (\mathbb{R} \times \mathbb{R})^*, J(q, p) = (q_1 \cdot \bar{p}_1^T, q_2 \cdot \bar{p}_2^T),$$

for any $(q, p) = (q_1, q_2, p_1, p_2) \in \mathbb{R}^4 \setminus \{0\} \times \mathbb{R}^4$ with $\bar{p}_i^T = (p_{i2}, -p_{i1})$, $i = 1, 2$ and $J^{-1}(\mathbb{R}^+ \mu) = \{(q, p) \in (\mathbb{R}^4 \setminus \{0\}) \times \mathbb{R}^4 \mid q_1 \cdot \bar{p}_1^T = 0, q_2 \cdot \bar{p}_2^T \in \mathbb{R}^+\}$. By Theorem 3.1, the ray reduced space $(T^*(\mathbb{R}^{2*} \times \mathbb{R}^{2*}))_{\mathbb{R}^+ \mu}$ is embedded in $T^*(\frac{\mathbb{R}^{2*} \times \mathbb{R}^{2*}}{\{e\} \times S^1}) \simeq T^*(\mathbb{R}^2 \setminus \{0\} \times (0, \infty))$. The reduced Rayleigh system is given by

$$H_{\mathbb{R}^+ \mu}(q_1, s_1, p_1, s_2) = \frac{1}{2}(\|q_1\|^2 + \|p_1\|^2 + \|s\|^2), \quad R_{\mathbb{R}^+ \mu}(q) = \|q_1\|^2 + \|p_1\|^2.$$

One can easily check that the only relative equilibrium points are given by $q_1 = (0, 0)$, $q_2 = (0, \alpha)$, $p_1 = (0, 0)$, $p_2 = (-\alpha, 0)$ with corresponding velocity $\xi = (0, 1) \in \mathfrak{g}_{\mathbb{R}^+ \mu} = \mathbb{R} \times \mathbb{R}$.

5. RAY REDUCTIONS OF COTANGENT AND COSPHERE BUNDLES OF A LIE GROUP

In this section we will determine the ray reduced spaces for lifted actions on cotangent and cosphere bundles. We will show that these ray reduced spaces are universal in the sense that any (symplectic) contact (ray) reduced space can be recovered from the (ray) reduced space of a (cotangent) or cosphere bundle.

Let G denote a d -dimensional Lie group with Lie algebra \mathfrak{g} . G acts on itself by left translations. This action lifts canonically to an action on T^*G which admits an equivariant and right invariant momentum map

$$(5.1) \quad J_L : T^*G \rightarrow \mathfrak{g}^* \quad , \quad J_L(\alpha_g) := T_e^* R_g(\alpha_g).$$

Similarly, for right translations we can construct the equivariant and left invariant momentum map

$$(5.2) \quad J_R : T^*G \rightarrow \mathfrak{g}^* \quad , \quad J_R(\alpha_g) := T_e^* L_g(\alpha_g).$$

Denote by \mathcal{O}_μ the coadjoint orbit of an element μ of \mathfrak{g}^* and let $\mathcal{O}_{\mathbb{R}^+ \mu}$ be its cone defined by (2.1).

Since for the ray-reduction the role of the coadjoint orbit will be played by a diagonal product of its cone and the quotient of G by the corresponding kernel group, we will now describe their manifold structure. We will see that, in general, $\mathcal{O}_{\mathbb{R}^+ \mu}$ is an immersed smooth submanifold of \mathfrak{g}^* .

Definition 5.1. Let $G_{\mathbb{R}^+ \mu}$ be the *ray isotropy group* of μ defined by $G_{\mathbb{R}^+ \mu} := \{g \in G \mid \text{Ad}_g^* \mu = r_g \mu, \text{ for } r_g \in \mathbb{R}^+\}$.

Lemma 5.1. *The ray isotropy group $G_{\mathbb{R}^+ \mu}$ is a closed Lie subgroup of G . Its Lie algebra is given by*

$$\mathfrak{g}_{\mathbb{R}^+ \mu} = \{\xi \in \mathfrak{g} \mid \text{ad}_\xi^* \mu = r_\xi \mu \quad \text{for a } r_\xi \in \mathbb{R}\}.$$

Proof. We have the following sequence of subgroups $G_\mu < G_{\mathbb{R}^+ \mu} < G$. To prove that the ray isotropy group is closed in G , suppose $(g_n)_{n \in \mathbb{N}}$ is a convergent sequence in $G_{\mathbb{R}^+ \mu}$ with $\lim_{n \rightarrow \infty} g_n = g \in G$. Then $\lim_{n \rightarrow \infty} \text{Ad}_{g_n}^* \mu = (\lim_{n \rightarrow \infty} r_{g_n}) \mu = \text{Ad}_g^* \mu$, for $(r_{g_n})_{n \in \mathbb{N}}$ a convergent sequence of positive numbers. Since the coadjoint map is linear and $\mu \neq 0$, $\lim_{n \rightarrow \infty} r_{g_n}$ is a strictly positive number and hence $g \in G_{\mathbb{R}^+ \mu}$. Thus, the ray

isotropy group is closed. To determine its Lie algebra, let first ξ be an element of $\mathfrak{g}_{\mathbb{R}^+ \mu}$. We want to show that $\exp(t\xi)$ belongs to $G_{\mathbb{R}^+ \mu}$ for arbitrary $t \in \mathbb{R}$. Then

$$\frac{d}{dt} \text{Ad}_{\exp t\xi}^* \mu = \text{Ad}_{\exp t\xi}^* (ad_{\xi}^* \mu) = \text{Ad}_{\exp t\xi}^* (r_{\xi} \mu) = r_{\xi} \text{Ad}_{\exp t\xi}^* \mu.$$

We have used the following formula

$$(5.3) \quad \frac{d}{dt} \text{Ad}_{g(t)}^* \mu(t) = \text{Ad}_{g(t)}^* \left(ad_{\xi(t)}^* \mu(t) + \frac{d\mu}{dt} \right),$$

where $\xi(t) = T_{g(t)} R_{g(t)}^{-1} \left(\frac{dg}{dt} \right)$ and $g(t)$, $\mu(t)$ are smooth curves in G and \mathfrak{g}^* , respectively. It follows that $\text{Ad}_{\exp t\xi}^* \mu = e^{r_{\xi} t} \mu$ and $\exp t\xi \in G_{\mathbb{R}^+ \mu}$, for every real t . For the reverse inclusion, suppose ξ is an element of the Lie algebra of the ray isotropy group. Then we know that $\exp t\xi \in G_{\mathbb{R}^+ \mu}$ and $\text{Ad}_{\exp t\xi}^* \mu = r_t \mu$ with r_t a positive real number for every $t \in \mathbb{R}$. Deriving at zero the above equality, we obtain that $ad_{\xi}^* \mu = \left(\frac{d}{dt} \Big|_{t=0} r_t \right) \mu$, completing thus the proof of this lemma. \square

Remark 5.1. As we saw in the proof of Theorem 2.1, if the Lie group G acts in a Hamiltonian way on the manifold M and this action admits an equivariant momentum map $J : M \rightarrow \mathfrak{g}^*$, then for every $x \in J^{-1}(\mathbb{R}^+ \mu)$ we have that

$$J^{-1}(\mathbb{R}^+ \mu) \cap (G \cdot x) = G_{\mathbb{R}^+ \mu} \cdot x \quad , \quad T_x(G_{\mathbb{R}^+ \mu} \cdot x) = T_x(G \cdot x) \cap T_x(J^{-1}(\mathbb{R}^+ \mu)).$$

Lemma 5.2. *The ray isotropy group $G_{\mathbb{R}^+ \mu}$ acts on $G \times \mathbb{R}^+$ by $g' \cdot (g, r) \rightarrow (g'g, \frac{r}{r_{g'}})$, where $\text{Ad}_{g'}^* \mu = r_{g'} \mu$. This action is free and proper and, therefore, the twisted product $G \times_{G_{\mathbb{R}^+ \mu}} \mathbb{R}^+$ is well defined. Even more, the surjective map*

$$f : G \times \mathbb{R}^+ \rightarrow \mathcal{O}_{\mathbb{R}^+ \mu} \quad , \quad f(g, r) := \text{Ad}_g^*(r\mu)$$

descends to a diffeomorphism on the twisted product $G \times_{G_{\mathbb{R}^+ \mu}} \mathbb{R}^+$.

Proof. Since it consists of direct calculations, we skip the proof of this Lemma. \square

Remark 5.2. Note that the above Lemma implies that the dimension of the cone coadjoint orbit at μ is given by $\dim \mathcal{O}_{\mathbb{R}^+ \mu} = \dim G + 1 - \dim G_{\mathbb{R}^+ \mu}$.

For technical reasons we need a precise description of the tangent space of the cone coadjoint orbit.

Lemma 5.3. *Let $\mathcal{O}_{\mathbb{R}^+ \mu}$ be the cone of the coadjoint orbit through $\mu \in \mathfrak{g}^*$. Then its tangent space at μ is given by*

$$T_{\mu} \mathcal{O}_{\mathbb{R}^+ \mu} = \{ad_{\xi}^* \mu + r\mu \mid r \in \mathbb{R}, \xi \in \mathfrak{g}\}.$$

Proof. Consider the smooth curve in $\mathcal{O}_{\mathbb{R}^+ \mu}$ given by $\mu(t) := \text{Ad}_{\exp(t\xi)}^*(e^{tr}\mu)$, where r is an arbitrary real number. Note that $\mu(0) = \mu$ and $\frac{d}{dt} \Big|_{t=0} \mu(t) = \xi_{\mathfrak{g}^*}(\mu) + r\mu = ad_{\xi}^* \mu + r\mu$. Therefore, $A := \{ad_{\xi}^* \mu + r\mu \mid r \in \mathbb{R}, \xi \in \mathfrak{g}\} \subset T_{\mu} \mathcal{O}_{\mathbb{R}^+ \mu}$.

Let $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}^+ \mu} \oplus \mathfrak{m}_{\mathbb{R}^+ \mu}$ be a splitting of \mathfrak{g} , and $\{\xi_1, \dots, \xi_k\}$, $\{\xi_{k+1}, \dots, \xi_d\}$ basis of $\mathfrak{g}_{\mathbb{R}^+ \mu}$ and $\mathfrak{m}_{\mathbb{R}^+ \mu}$, respectively. It is easy to see that the set $\{\xi_{k+1} \mathfrak{g}^*(\mu), \dots, \xi_d \mathfrak{g}^*(\mu), \mu\}$ forms a basis of A . And since $\dim(\{\xi_{k+1} \mathfrak{g}^*(\mu), \dots, \xi_d \mathfrak{g}^*(\mu), \mu\})$ is $d+1 - \dim(G_{\mathbb{R}^+ \mu})$, it follows that $A = T_{\mu} \mathcal{O}_{\mathbb{R}^+ \mu}$. \square

Proposition 5.1. *The cone coadjoint orbit $\mathcal{O}_{\mathbb{R}^+ \mu}$ is an initial Poisson submanifold of \mathfrak{g}^* and if the coadjoint action is proper, it is even a closed embedded submanifold.*

Proof. The simplest way to see this is to notice that the ray coadjoint orbit of μ is actually the orbit through μ of the following action of $G \times \mathbb{R}^+$ on \mathfrak{g}^*

$$(g, r) \cdot \mu' := Ad_{g^{-1}}^* r \mu',$$

for any $(g, r) \in G \times \mathbb{R}^+$ and any $\mu' \in \mathfrak{g}^*$. Therefore, as any orbit it is an initial Poisson submanifold. Since the coadjoint action of G is proper, so is the action of $G \times \mathbb{R}^+$. Therefore, $\mathcal{O}_{\mathbb{R}^+\mu}$ is a closed embedded submanifold of \mathfrak{g}^* . Of course, one can easily verify that the smooth structure of $\mathcal{O}_{\mathbb{R}^+\mu}$ as orbit of the $(G \times \mathbb{R}^+)$ -action coincides with the one described in Lemma 5.2. Indeed, $G \times_{G \times \mathbb{R}^+} \mathbb{R}^+$ and $\frac{G \times \mathbb{R}^+}{(G \times \mathbb{R}^+)_{\mu}}$ are diffeomorphic manifolds. \square

Since the coadjoint action of G restricts to $\mathcal{O}_{\mathbb{R}^+\mu}$, we have the following.

Fix μ an element of \mathfrak{g}^* . Notice that the Lie algebra of the kernel group of μ , \mathfrak{k}_{μ} is closed in \mathfrak{g}_{μ} . Let U be an open neighborhood of 0 in \mathfrak{g} such that the exponential map $\exp : U \rightarrow \exp(U)$ is a diffeomorphism. Choose $V \subset U$ a closed neighborhood of 0. Then $\exp(V) \subset \exp(U)$ is a closed neighborhood of e . We want to show that $\exp(V) \cap K_{\mu}$ is closed in G . Thus, suppose $(k_n)_{n \in \mathbb{N}} = (\exp \xi_n)_{n \in \mathbb{N}}$ is a convergent sequence of $\exp(V) \cap K_{\mu}$ with $(\xi_n)_{n \in \mathbb{N}}$ a sequence in V . Since $\exp^{-1} k_n = \xi_n$ for every $n \in \mathbb{N}$, it follows that in fact $\xi_n \in \mathfrak{k}_{\mu}$. Using the continuity of the exponential map and the fact that the kernel algebra is closed in \mathfrak{g} , we have that $\lim_{n \rightarrow \infty} \exp^{-1} k_n = \lim_{n \rightarrow \infty} \xi_n = \xi \in \mathfrak{k}_{\mu}$. Therefore $\lim_{n \rightarrow \infty} \exp \xi_n = \exp \lim_{n \rightarrow \infty} \xi_n = \exp \xi \in K_{\mu}$ and $\exp(V) \cap K_{\mu}$ is closed in G . A standard result of Lie theory (see, for instance, [10]), Corollary 1.10.7) implies that the kernel group of μ is a closed regular Lie subgroup of G and the quotient $\frac{G}{K_{\mu}}$ is a smooth manifold.

Now we are ready to define the manifold which will play the role of the cotangent orbit for the ray reduction, namely the diagonal product of the cone coadjoint orbit and the quotient of G by the corresponding kernel group

$$\text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_{\mu}} \right) := \{ (Ad_g^* r \mu, \hat{g}) \mid g \in G \text{ and } r \in \mathbb{R}^+ \}.$$

Recall that given two surjective submersions $\pi_1 : M_1 \rightarrow E$ and $\pi_2 : M_2 \rightarrow E$, the diagonal of $M_1 \times M_2$ over (π_1, π_2) , $\text{Diag}(M_1 \times M_2) := \{(x_1, x_2) \in M_1 \times M_2 \mid \pi_1(x_1) = \pi_2(x_2)\}$ is a submanifold of $M_1 \times M_2$ and its tangent space is given by

$$T_{(x_1, x_2)}(\text{Diag}(M_1 \times M_2)) \simeq \{(v_1, v_2) \in T_{x_1} M_1 \times T_{x_2} M_2 \mid T_{x_1} \pi_1(v_1) = T_{x_2} \pi_2(v_2)\} = \text{Diag}(T_{x_1} M_1 \times T_{x_2} M_2).$$

In particular, for $\pi_1 : \mathcal{O}_{\mathbb{R}^+\mu} \rightarrow \frac{G}{K_{\mu}}$ defined by $\pi_1(Ad_g^* r \mu) := \hat{g}$ and π_2 the canonical projection from $\frac{G}{K_{\mu}}$ onto $\frac{G}{G_{\mathbb{R}^+\mu}}$ we obtain that

$$T_{(Ad_g^* r \mu, \hat{g})} \text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_{\mu}} \right) \simeq \text{Diag} \left(T_{Ad_g^* r \mu} \mathcal{O}_{\mathbb{R}^+\mu}, T_{\hat{g}} \frac{G}{K_{\mu}} \right) \simeq T_{Ad_g^* r \mu} \mathcal{O}_{\mathbb{R}^+\mu}.$$

More precisely, we have that

$$T_{(Ad_g^* r \mu, \hat{g})} \text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_{\mu}} \right) = \{ (ad_{\xi}^*(Ad_g^* r \mu) + r' Ad_g^* r \mu, \hat{\xi}_G(\hat{g})) \mid \xi \in \mathfrak{g}, r' \in \mathbb{R} \},$$

for any $(Ad_g^* r \mu, \hat{g}) \in \mathcal{O}_{\mathbb{R}^+\mu}$. Here $\hat{\xi}_G(\hat{g})$ denotes the projection on $\frac{G}{K_{\mu}}$ of the infinitesimal isometry associated to ξ with respect to the action by left translations of

G on itself. Let $\omega_{\mathbb{R}^+\mu}^-$ be the two form on $\text{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$ defined by

$$(5.4) \quad \begin{aligned} \omega_{\mathbb{R}^+\mu}^-(Ad_g^*r\mu, \hat{g}) & ((ad_{\xi_1}^* Ad_g^*r\mu + r_1 Ad_g^*r\mu, \xi_{1G}(\hat{g})), (ad_{\xi_2}^* Ad_g^*r\mu + r_2 Ad_g^*r\mu, \xi_{2G}(\hat{g}))) \\ &= -\langle Ad_g^*r\mu, [\xi_1, \xi_2] \rangle + r_2 \langle Ad_g^*r\mu, \xi_1 \rangle - r_1 \langle Ad_g^*r\mu, \xi_2 \rangle, \end{aligned}$$

for any $(Ad_g^*r\mu, \hat{g}) \in \text{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$ and any tangent vectors $(ad_{\xi_i}^* Ad_g^*r\mu + r_i Ad_g^*r\mu, \xi_{iG}(\hat{g}))_{i=1,2} \in T_{(Ad_g^*r\mu, \hat{g})} \text{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$. In fact, as we will see from Theorem 5.1, $\left(\text{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right), \omega_{\mathcal{O}_{\mathbb{R}^+\mu}}^-\right)$ is a well defined symplectic manifold. One could also prove this directly, but we prefer to skip the computations and apply instead the ray reduction.

Theorem 5.1. *Consider the cotangent lift of the action by left translations of a Lie group G on itself. For every $\mu \in \mathfrak{g}^*$ with $\ker \mu + \mathfrak{g}_\mu = \mathfrak{g}$, the ray reduced space $(T^*(G)_{\mathbb{R}^+\mu}, \omega_{\mathbb{R}^+\mu})$ is well defined and symplectomorphic to the diagonal manifold $\left(\text{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right), \omega_{\mathbb{R}^+\mu}^-\right)$ with symplectic form $\omega_{\mathbb{R}^+\mu}^-$ defined by (5.4).*

Proof. Since the cotangent lift of left translations is a free and proper action, if μ is an element of \mathfrak{g}^* with $\ker \mu + \mathfrak{g}_\mu = \mathfrak{g}$ the ray reduced space at μ , $(T^*G)_{\mathbb{R}^+\mu} = \frac{J_L^{-1}(\mathbb{R}^+\mu)}{K_\mu}$ is well defined. J_R is the momentum map defined by (5.1).

Note that $J_L^{-1}(\mathbb{R}^+\mu) = \{T_g^*R_{g^{-1}}(r\mu) \mid g \in G, r \in \mathbb{R}^+\}$. The momentum map associated to right translations (see (5.2)) induces the application $\bar{J}_R : (T^*G)_{\mathbb{R}^+\mu} \rightarrow \text{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right)$ defined by $\bar{J}_R([\alpha_g]) := (J_R(\alpha_g), \hat{g}) = (Ad_g^*r\mu, \hat{g})$, for any $\alpha_g = T_g^*R_{g^{-1}}(r\mu)$. To see that \bar{J}_R is well defined, fix an arbitrary $k \in K_\mu$. Then,

$$\begin{aligned} \bar{J}_R([k \cdot \alpha_g]) &= \bar{J}_R([k \cdot T_g^*R_{g^{-1}}r\mu]) = \bar{J}_R([T_{kg}^*L_{k^{-1}}T_g^*R_{g^{-1}}r\mu]) \\ &= (T_e^*L_{kg}T_{kg}^*(R_{g^{-1}} \circ L_{k^{-1}})(r\mu), \hat{k}g) = (Ad_g^*(r\mu), \hat{g}), \end{aligned}$$

proving thus that \bar{J}_R is indeed well-defined. Since the kernel group of μ is a subgroup of its isotropy group, \bar{J}_R is also one to one. Surjectiveness is obvious and hence \bar{J}_R is a bijection. Its inverse is given by $\bar{J}_{R^{-1}} : \text{Diag}\left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu}\right) \rightarrow (T^*G)_{\mathbb{R}^+\mu}$, $\bar{J}_{R^{-1}}(Ad_g^*r\mu, \hat{g}) = [T_g^*R_{g^{-1}}r\mu]$.

To prove that \bar{J}_R is smooth, and hence a diffeomorphism we will use the right invariant 1-form $\lambda \in \Lambda^1(G)$ given by $\lambda(g)(v_g) := T_g^*R_{g^{-1}}\mu(v_g)$. The graph of λ defines the diffeomorphism $F' : G \rightarrow J_L^{-1}(\mu)$, $F'(g) := \lambda(g) = T_g^*R_{g^{-1}}\mu$. Consider the map $F : G \times \mathbb{R}^+ \rightarrow J_L^{-1}(\mathbb{R}^+\mu)$ given by $F(g, r) := F'(g)r$, for any elements $g \in G$ and $r \in \mathbb{R}^+$. It is obviously smooth and we want to show that it descends to a diffeomorphism $\bar{F} : \text{Diag}\left(G \times_{G_{\mathbb{R}^+\mu}} \mathbb{R}^+ \times \frac{G}{K_\mu}\right) \rightarrow (T^*G)_{\mathbb{R}^+\mu}$, $\bar{F}([g, r], \hat{g}) = [T_g^*R_{g^{-1}}r\mu]$. Let us first verify that it is a well defined map. For this, let $(k, r_k) \in G \times_{G_{\mathbb{R}^+\mu}} \mathbb{R}^+$ with $\hat{g} = \hat{k}g$, so that $[(kg, \frac{r}{r_k}), \hat{k}g] = [(g, r), \hat{g}]$. The equality of the second components implies that actually k belongs to the kernel group of μ . Then

we obtain

$$\begin{aligned}\bar{F}([(kg, \frac{r}{r_k}), \hat{k}g]) &= [T_{kg}^* R_{(kg)^{-1}} \frac{r}{r_k} \mu] = [T_{kg}^* R_{g^{-1}} T_k^* L_{k^{-1}} T_e^* L_k T_k^* R_{k^{-1}} \frac{r}{r_k} \mu] = \\ &= [T_{kg}^* R_{g^{-1}} T_k^* L_{k^{-1}} Ad_k^* \frac{r}{r_k} \mu] = [k \cdot T_g^* R_{g^{-1}} r \mu] = [T_g^* R_{g^{-1}} r \mu] = \bar{F}([g, r], \hat{g}).\end{aligned}$$

Observe that $\bar{J}_R \circ \bar{F}$ is precisely the diffeomorphism of Lemma 5.2. Therefore, \bar{J}_R is also a diffeomorphism. Its inverse is given by

$$\bar{J}_{R^{-1}} : \text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu} \right) \rightarrow (T^*G)_{\mathbb{R}^+\mu} \quad , \quad \bar{J}_{R^{-1}}(Ad_g^* r \mu, \hat{g}) = [T_g^* R_{g^{-1}} r \mu],$$

and we can endow $\text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu} \right)$ with the symplectic form $\omega_{\bar{\mathcal{O}}_{\mathbb{R}^+\mu}}^- := \bar{J}_{R^{-1}}^* \omega_{\mathbb{R}^+\mu}$.

In order to give the explicit description of $\omega_{\bar{\mathcal{O}}_{\mathbb{R}^+\mu}}^-$ fix $(Ad_g^* r \mu, \hat{g}) \in \text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu} \right)$

and two tangent vectors $\{v_i = (ad_{\xi_i}^* Ad_g^* r \mu + r_i Ad_g^* r \mu, \hat{\xi}_{iG}(\hat{g}))\}_{i=1,2}$ in

$T_{(Ad_g^* r \mu, \hat{g})} \text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu} \right)$. It follows that

$$\omega_{\bar{\mathcal{O}}_{\mathbb{R}^+\mu}}^-(Ad_g^* r \mu, \hat{g})(v_1, v_2) = \omega_{\mathbb{R}^+\mu}([T_g^* R_{g^{-1}} r \mu])(T_{(Ad_g^* r \mu, \hat{g})} \bar{J}_{R^{-1}}(v_1), T_{(Ad_g^* r \mu, \hat{g})} \bar{J}_{R^{-1}}(v_2)).$$

Note that

$$\begin{aligned}T_{(Ad_g^* r \mu, \hat{g})} \bar{J}_{R^{-1}}(v_i) &= \\ &= T_{(Ad_g^* r \mu, \hat{g})} \bar{J}_{R^{-1}} \left(\frac{d}{dt} \Big|_{t=0} \left(Ad_{\exp t \xi_i}^* e^{tr_i} Ad_g^* r \mu, (\exp t \xi_i \cdot g) \right) \right) \\ &= \frac{d}{dt} \Big|_{t=0} \left(\bar{J}_{R^{-1}}(Ad_{\exp t \xi_i}^* e^{tr_i} Ad_g^* r \mu, (\exp t \xi_i \cdot g)) \right) \\ &= \frac{d}{dt} \Big|_{t=0} \left(\pi_{K_\mu}(T_{g \exp t \xi_i}^* R_{(g \exp t \xi_i)^{-1}} e^{tr_i} \mu) \right) \\ &= T_{T_g^* R_{g^{-1}} r \mu} \pi_{K_\mu} \left(\frac{d}{dt} \Big|_{t=0} e^{tr_i} (T_g^* R_{g^{-1}} r \mu) \cdot \exp t \xi_i \right) = T_{T_g^* R_{g^{-1}} r \mu} \pi_{K_\mu}(X^{\xi_i}(T_g^* R_{g^{-1}} r \mu)),\end{aligned}$$

where X^{ξ_i} is the vector field on T^*G with flow given by

$\Phi^i(t, \alpha_{g'}) := T_{g \exp t \xi_i}^* R_{\exp -t \xi_i} e^{tr_i} \alpha_{g'}$, for any $\alpha_{g'} \in T_{g'}^*G$. Then, using the fact that $\pi_{\mathbb{R}^+\mu}^* \omega_{\mathbb{R}^+\mu} = i_{\mathbb{R}^+\mu}^*(-d\theta)$ and the above calculus, we obtain

$$\begin{aligned}\omega_{\bar{\mathcal{O}}_{\mathbb{R}^+\mu}}^-(Ad_g^* r \mu, \hat{g})(v_1, v_2) &= \pi_{K_\mu}^* \omega_{\mathbb{R}^+\mu}(T_g^* R_{g^{-1}} r \mu)(X^{\xi_1}(T_g^* R_{g^{-1}} r \mu), X^{\xi_2}(T_g^* R_{g^{-1}} r \mu)) \\ &= -d\theta(T_g^* R_{g^{-1}} r \mu)(X^{\xi_1}(T_g^* R_{g^{-1}} r \mu), X^{\xi_2}(T_g^* R_{g^{-1}} r \mu)) = \\ &= -X^{\xi_1}(\theta(X^{\xi_2})(T_g^* R_{g^{-1}} r \mu) + X^{\xi_2}(\theta(X^{\xi_1})(T_g^* R_{g^{-1}} r \mu) + \theta([X^{\xi_1}, X^{\xi_2}])(T_g^* R_{g^{-1}} r \mu)).\end{aligned}$$

Next, we want to show that

$$(5.5) \quad \theta(X^{\xi_i}) = J_R^{\xi_i} \text{ and } X^{\xi_i}(J_R^{\xi_j})(T_g^* R_{g^{-1}} r \mu) = \langle Ad_g^* r \mu, [\xi_i, \xi_j] \rangle + r_i \langle Ad_g^* r \mu, \xi_j \rangle,$$

for $i = 1, 2$. Indeed, for any $\alpha_g \in T_g^*G$ we have

$$\begin{aligned}\theta(X^{\xi_i})(\alpha_g) &= \langle \alpha_g, T_{\alpha_g} \pi(X^{\xi_i}(\alpha_g)) \rangle = \langle \alpha_g, \frac{d}{dt} \Big|_{t=0} \pi(e^{tr_i} \alpha_g \cdot \exp t \xi_i) \rangle \\ &= \langle \alpha_g, \xi_{iG}(g) \rangle = J_R^{\xi_i}(\alpha_g).\end{aligned}$$

This also implies that X^{ξ_i} and ξ_{iG} are π -related vector fields. And

$$\begin{aligned} X^{\xi_i}(J_R^{\xi_j})(T_g^* R_{g^{-1}} r\mu) &= \left. \frac{d}{dt} \right|_{t=0} J_R^{\xi_j}(e^{tr_i} T_g^* R_{g^{-1}} r\mu \cdot \exp t\xi_i) = \\ &= \left. \frac{d}{dt} \right|_{t=0} T_e^* L_{g \exp t\xi_i}(T_{g \exp t\xi_i}^* R_{\exp -t\xi_i}(e^{tr_i} T_g^* R_{g^{-1}} r\mu))(\xi_j) = \\ &= \left. \frac{d}{dt} \right|_{t=0} Ad_{g \exp t\xi_i}^*(e^{tr_i} r\mu)(\xi_j) = Ad_g^*(ad_{Ad_g \xi_i}^* r\mu + r_i r\mu)(\xi_j) = \\ &= Ad_g^*(ad_{Ad_g \xi_i}^* r\mu + r_i r\mu)(\xi_j) = (ad_{\xi_i}^*(Ad_g^* r\mu) + r_i Ad_g^* r\mu)(\xi_j). \end{aligned}$$

Note that in the above calculation we have again used formula (5.3). Applying (5.5), it follows that

$$\begin{aligned} \omega_{\mathcal{O}_{\mathbb{R}^+ \mu}}^-(Ad_g^* r\mu, \hat{g})(v_1, v_2) &= -X^{\xi_1}(\theta(X^{\xi_2})(T_g^* R_{g^{-1}} r\mu) + X^{\xi_2}(\theta(X^{\xi_1})(T_g^* R_{g^{-1}} r\mu) \\ &\quad + \theta(X^{[\xi_1, \xi_2]})(T_g^* R_{g^{-1}} r\mu) = -\langle Ad_g^* r\mu, [\xi_1, \xi_2] \rangle - r_1 \langle Ad_g^* r\mu, \xi_2 \rangle \\ &\quad + \langle Ad_g^* r\mu, [\xi_2, \xi_1] \rangle + r_2 \langle Ad_g^* r\mu, \xi_1 \rangle + J_R^{[\xi_1, \xi_2]}(T_g^* R_{g^{-1}} r\mu) \\ &\quad = -\langle Ad_g^* r\mu, [\xi_1, \xi_2] \rangle + r_2 \langle Ad_g^* r\mu, \xi_1 \rangle - r_1 \langle Ad_g^* r\mu, \xi_2 \rangle. \end{aligned}$$

In particular, for $g = e$ and $r = 1$ we have that

$$\begin{aligned} \omega_{\mathcal{O}_{\mathbb{R}^+ \mu}}^-(\mu, \hat{e})(ad_{\xi_1}^* \mu + r_1 \mu, \hat{\xi}_1), (ad_{\xi_2}^* \mu + r_2 \mu, \hat{\xi}_2)) &= -\langle \mu, [\xi_1, \xi_2] \rangle + r_2 \langle \mu, \xi_1 \rangle \\ &\quad - r_1 \langle \mu, \xi_2 \rangle, \text{ for any } \xi_{\{i=1,2\}} \in \mathfrak{g}. \end{aligned}$$

The first term in the above expression is precisely $\omega_{\mathcal{O}_{\mathbb{R}^+ \mu}}^-(\mu)(ad_{\xi_1}^* \mu, ad_{\xi_2}^* \mu)$ and hence the minus sign in the notation of the symplectic form on $\text{Diag}\left(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu}\right)$. \square

Corollary 5.1. *In the hypothesis of Theorem 5.1, the symplectic form $\omega_{\mathcal{O}_{\mathbb{R}^+ \mu}}^-$ defined by (5.4) is G -invariant with respect to the following action*

$$g_1 \cdot (Ad_g^* r\mu, \hat{g}) := \left(Ad_{g_1}^* Ad_g^* r\mu, \widehat{gg_1^{-1}} \right),$$

for each g_1 in G and $(Ad_g^* r\mu, \hat{g})$ in $\text{Diag}\left(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu}\right)$.

Proof. Fix g_1 in G and $x := (Ad_g^* r\mu, \hat{g})$ in $\text{Diag}\left(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu}\right)$. Let v_ξ be the tangent vector $(ad_\xi^*(Ad_g^* r\mu) + r_\xi Ad_g^* r\mu, \hat{\xi}_G(\hat{g})) \in T_{(Ad_g^* r\mu, \hat{g})} \text{Diag}\left(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu}\right)$. Here ξ is

an arbitrary element of \mathfrak{g} . Then, we have

$$\begin{aligned}
\omega_{\overline{\mathcal{O}}_{\mathbb{R}^+\mu}}(g_1 \cdot x)(g_1 \cdot v_\xi, g_1 \cdot v_\eta) &= \\
\omega_{\overline{\mathcal{O}}_{\mathbb{R}^+\mu}}(g_1 \cdot x) \left(\frac{d}{dt} \Big|_{t=0} (Ad_{\exp t\xi g_1^{-1}}^* Ad_g^* r\mu), T_g \widehat{R_{g_1^{-1}} \xi}_G(\hat{g}), \right. \\
&\quad \left. \frac{d}{dt} \Big|_{t=0} (Ad_{\exp t\eta g_1^{-1}}^* Ad_g^* r\mu), T_g \widehat{R_{g_1^{-1}} \eta}_G(\hat{g}) \right) = \\
\omega_{\overline{\mathcal{O}}_{\mathbb{R}^+\mu}}(g_1 \cdot x) \left(\left(Ad_{g_1^{-1}}^* v_\xi, (\widehat{Ad_{g_1} \xi})_G(\widehat{gg_1^{-1}}) \right), \left(Ad_{g_1^{-1}}^* v_\eta, (\widehat{Ad_{g_1} \eta})_G(\widehat{gg_1^{-1}}) \right) \right) &= \\
\omega_{\overline{\mathcal{O}}_{\mathbb{R}^+\mu}}(g_1 \cdot x) \left(\left(v_{Ad_{g_1} \xi}, (\widehat{Ad_{g_1} \xi})_G(\widehat{gg_1^{-1}}) \right), \left(v_{Ad_{g_1} \eta}, (\widehat{Ad_{g_1} \eta})_G(\widehat{gg_1^{-1}}) \right) \right) &= \\
- \langle Ad_{g_1^{-1}}^* r\mu, [Ad_{g_1} \xi, Ad_{g_1} \eta] \rangle + r_\eta \langle Ad_{g_1^{-1}}^* r\mu, Ad_{g_1} \xi \rangle - r_\xi \langle Ad_{g_1^{-1}}^* r\mu, Ad_{g_1} \eta \rangle &= \\
- \langle Ad_g^* r\mu, [\xi, \eta] \rangle + r_\eta \langle Ad_g^* r\mu, \xi \rangle - r_\xi \langle Ad_g^* r\mu, \eta \rangle = \omega_{\overline{\mathcal{O}}_{\mathbb{R}^+\mu}}(x)(v_\xi, v_\eta).
\end{aligned}$$

Therefore, $\omega_{\overline{\mathcal{O}}_{\mathbb{R}^+\mu}}$ is G -invariant. \square

Proposition 5.2. *The symplectomorphic G -action on $\left(\text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu} \right), \omega_{\overline{\mathcal{O}}_{\mathbb{R}^+\mu}} \right)$ admits an equivariant momentum map*

$$-I_{\mathcal{O}_{\mathbb{R}^+\mu}} : \text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu} \right) \rightarrow \mathfrak{g}^*, I_{\mathcal{O}_{\mathbb{R}^+\mu}}(Ad_g^* r\mu, \hat{g}) := -Ad_g^* r\mu,$$

for each $(Ad_g^* r\mu, \hat{g})$ in $\text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu} \right)$.

Proof. Let ξ be an element of \mathfrak{g} and denote by $I_{\mathcal{O}_{\mathbb{R}^+\mu}}^\xi : \text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu} \right) \rightarrow \mathbb{R}$ the map given by $(Ad_g^* r\mu, \hat{g}) \mapsto \langle Ad_g^* r\mu, \xi \rangle$. The infinitesimal generator associated to ξ on $\text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu} \right)$ is

$$\begin{aligned}
\xi_{\text{Diag}(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu})}(Ad_g^* r\mu, \hat{g}) &= \frac{d}{dt} \Big|_{t=0} \left(Ad_{\exp(-t\xi)}^* Ad_g^* r\mu, g \exp(-t\xi) \right) = \\
(ad_{-\xi}^*(Ad_g^* r\mu), -\hat{\xi}_G(\hat{g})), &\text{ for any } (Ad_g^* r\mu, \hat{g}) \in \text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu} \right).
\end{aligned}$$

Then, for all $(ad_\eta^* Ad_g^* r\mu + r_\eta Ad_g^* r\mu, \hat{\eta}_G(\hat{g})) \in T_{(Ad_g^* r\mu, \hat{g})} \text{Diag} \left(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu} \right)$, we obtain that

$$\begin{aligned}
(5.6) \quad i_{\xi_{\text{Diag}(\mathcal{O}_{\mathbb{R}^+\mu} \times \frac{G}{K_\mu})}} \omega_{\overline{\mathcal{O}}_{\mathbb{R}^+\mu}}(Ad_g^* r\mu, \hat{g})(ad_\eta^* Ad_g^* r\mu + r_\eta Ad_g^* r\mu, \hat{\eta}_G(\hat{g})) &= \\
\omega_{\overline{\mathcal{O}}_{\mathbb{R}^+\mu}}(Ad_g^* r\mu, \hat{g}) \left((ad_{-\xi}^*(Ad_g^* r\mu), -\hat{\xi}_G(\hat{g})), (ad_\eta^* Ad_g^* r\mu + r_\eta Ad_g^* r\mu, \hat{\eta}_G(\hat{g})) \right) &= \\
\langle Ad_g^* r\mu, [\xi, \eta] \rangle - r_\eta \langle Ad_g^* r\mu, \xi \rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(5.7) \quad T_{(Ad_g^* r\mu, \hat{g})} I_{\mathcal{O}_{\mathbb{R}^+\mu}}^\xi (ad_\eta^* Ad_g^* r\mu + r_\eta Ad_g^* r\mu, \hat{\eta}_G(\hat{g})) &= \frac{d}{dt} \Big|_{t=0} \langle Ad_{\exp t\eta}^* e^{tr\xi} Ad_g^* r\mu, \xi \rangle \\
&= -\langle Ad_g^* r\mu, [\xi, \eta] \rangle + r_\eta \langle Ad_g^* r\mu, \xi \rangle.
\end{aligned}$$

Equalities (5.6) and (5.7) imply that $X_{-J_{\mathcal{O}_{\mathbb{R}^+ \mu}}^\xi} = \xi_{\text{Diag}(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu})}$ for all $\xi \in \mathfrak{g}$. Hence the proof of this proposition is complete. \square

Recall that the symplectic difference of two symplectic manifolds $(M_i, \omega_i)_{i=1,2}$ is $M_1 \ominus M_2 := (M_1 \times M_2, \pi_1^* \omega_1 - \pi_2^* \omega_2)$, where $(\pi_i : M_1 \times M_2 \rightarrow M_i)_{i=1,2}$ are the canonical projections. If the Lie group G acts on both M_1 and M_2 such that these actions admit equivariant momentum maps $(J_i : M_i \rightarrow \mathfrak{g}^*)_{i=1,2}$, then the diagonal action of G on the symplectic difference $M_1 \ominus M_2$ admits an equivariant momentum map given by $J_d := J_1 \circ \pi_1 - J_2 \circ \pi_2 : M_1 \ominus M_2 \rightarrow \mathfrak{g}^*$.

The following theorem illustrates the theoretical importance of the diagonal product $\text{Diag}(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu})$ in the reduction procedure. Namely, any ray reduced space can be seen as the symplectic difference of the initial manifold and the diagonal product of the associated ray coadjoint orbit with the quotient of G by the kernel group.

Theorem 5.2 (Shifting Theorem). *Let the Lie group G act smoothly on the symplectic manifold (M, ω) such that it admits an equivariant momentum map $J : M \rightarrow \mathfrak{g}^*$. Fix μ an element of the dual Lie algebra of G and suppose that the hypothesis of Theorem 2.1 are fulfilled. Then G acts diagonally on $M \ominus \text{Diag}(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu})$ and its symplectic reduced space at zero is well defined. Even more, $(M \ominus \text{Diag}(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu}))_0$ is symplectomorphic to $M_{\mathbb{R}^+ \mu}$, the ray reduced space at μ of M .*

Proof. The symplectic difference $M \ominus \text{Diag}(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu})$ has symplectic form $\pi_1^* \omega - \pi_2^* \omega_{\mathcal{O}_{\mathbb{R}^+ \mu}}$ and momentum map $J_d := J \circ \pi_1 + I_{\mathcal{O}_{\mathbb{R}^+ \mu}} \circ \pi_2$. Of course, $\pi_1 : M \ominus \text{Diag}(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu}) \rightarrow M$ and $\pi_2 : M \ominus \text{Diag}(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu}) \rightarrow \text{Diag}(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu})$ are the canonical projections. It is easy to check that in the hypothesis of Theorem 2.1, the 0-symplectic reduced space is well defined.

Let $\phi : J^{-1}(\mathbb{R}^+ \mu) \rightarrow M \ominus \text{Diag}(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu})$ be the map defined by $x \in J^{-1}(\mathbb{R}^+ \mu) \mapsto (x, (-J(x), \hat{e}))$. Denote by $[\phi]$ its (K_μ, G) -projection

$$[\phi] : M_{\mathbb{R}^+ \mu} \rightarrow \left(M \ominus \text{Diag} \left(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu} \right) \right)_0, [\phi](\hat{x}) := [x, (-J(x), \hat{e})],$$

where $[\cdot]$ and $\widehat{\cdot}$ denote the G and K_μ -classes, respectively. This map is well defined. Indeed, let k be an element of the kernel group of μ . Then, $[\phi](\widehat{kx}) = [kx, (-J(kx), \hat{e})] = [kx, (-k \cdot J(x), \widehat{k^{-1}})] = [k \cdot (x, (-J(x), \hat{e}))] = [\phi](\hat{x})$, for any $\hat{x} \in M_{\mathbb{R}^+ \mu}$. To see that $[\phi]$ is injective, let \hat{x}_1, \hat{x}_2 be elements of $M_{\mathbb{R}^+ \mu}$ such that $[x_1, (-J(x_1), \hat{e})] = [x_2, (-J(x_2), \hat{e})]$. Then, there is g an element of G such that $(gx_1, (-gJ(x), \hat{g}^{-1})) = (x_2, (-J(x_2), \hat{e}))$. It follows that $g \in K_\mu$ and $gx_1 = x_2$. Hence $\hat{x}_1 = \hat{x}_2$ and $[\phi]$ is one-to-one. If $[x, (Ad_g^* r \mu, \hat{g})]$ is an element of $(M \ominus \text{Diag}(\mathcal{O}_{\mathbb{R}^+ \mu} \times \frac{G}{K_\mu}))_0$, then $J_d(x, (Ad_g^* r \mu, \hat{g})) = J(x) + Ad_g^* r \mu = 0$. Therefore, $gx \in J^{-1}(\mathbb{R}^+ \mu)$,

$$[\phi](\hat{gx}) = [gx, (-J(gx), \hat{e})] = [gx, (-gAd_g^* r \mu, \widehat{gg^{-1}})] = [x, (-J(x), \hat{e})],$$

and $[\phi]$ is onto. As it is obviously a smooth map, we obtain that $[\phi]$ is in fact a diffeomorphism with inverse given by $[x, (Ad_g^* r \mu, \hat{g})] \mapsto [gx]$.

To show that $[\phi]$ is also a symplectic map, fix \hat{x} in $M_{\mathbb{R}^+\mu}$ and $(v_i)_{i=1,2}$ in $T_x J^{-1}(\mathbb{R}^+\mu)$. Note that $T_x(\pi_2 \circ \phi)(v_i)$ belongs to $\mathbb{R}\mu \simeq T_{J(x)}(\mathbb{R}^+\mu)$ for each $i = 1, 2$. Suppose $J(x) = r\mu$ and $(T_x(\pi_2 \circ \phi)(v_i) = r_i\mu)_{i=1,2}$ with $(r_i)_{i=1,2}$ reals. Then, using the function equalities $[\phi] \circ \pi_{K_\mu} = \pi_G \circ \phi$ and $\pi_1 \circ \phi = Id_{J^{-1}(\mathbb{R}^+\mu)}$, we obtain

$$\begin{aligned} & [\phi]^*(\pi_1^*\omega - \pi_2^*\omega_{\mathcal{O}_{\mathbb{R}^+\mu}}^-)_0(\hat{x})(T_x\pi_{K_\mu}v_1, T_x\pi_{K_\mu}v_2) = \\ & (\pi_1^*\omega - \pi_2^*\omega_{\mathcal{O}_{\mathbb{R}^+\mu}}^-)_0([x, (-J(x), \hat{e})])(T_x([\phi] \circ \pi_{K_\mu})v_1, T_x([\phi] \circ \pi_{K_\mu})v_2) = \\ & (\pi_1^*\omega - \pi_2^*\omega_{\mathcal{O}_{\mathbb{R}^+\mu}}^-)(\phi(x))(T_x\phi v_1, T_x\phi v_2) = \\ & \omega(x)(T_x(\pi_1 \circ \phi)v_1, T_x(\pi_1 \circ \phi)v_2) - \omega_{\mathcal{O}_{\mathbb{R}^+\mu}}^-(J(x), \hat{e})(T_x(\pi_2 \circ \phi)v_1, T_x(\pi_2 \circ \phi)v_2) = \\ & i_\mu^*\omega(x)(T_x(i_\mu \circ \pi_1 \circ \phi)v_1, T_x(i_\mu \circ \pi_1 \circ \phi)v_2) - \omega_{\mathcal{O}_{\mathbb{R}^+\mu}}^-(r\mu, \hat{e})(r_1\mu, r_2\mu) = \\ & i_\mu^*\omega(x)(T_x(i_\mu \circ \pi_1 \circ \phi)v_1, T_x(i_\mu \circ \pi_1 \circ \phi)v_2) = \omega_{\mathbb{R}^+\mu}(\hat{x})(T_x\pi_{K_\mu}v_1, T_x\pi_{K_\mu}v_2), \end{aligned}$$

completing thus the proof of this theorem. \square

In the remaining of this section we will study the ray reduced spaces of the cosphere bundle of the Lie group G . Consider the action of the multiplicative group \mathbb{R}^+ by dilatations on the fibers of $T^*G \setminus \{0_{T^*G}\}$. The cosphere bundle of G , S^*G is the quotient manifold $(T^*G \setminus \{0_{T^*G}\})/\mathbb{R}^+$. Denote by $\pi : T^*G \setminus \{0_{T^*G}\} \rightarrow S^*(G)$ the canonical projection. Then, $(\pi, \mathbb{R}^+, T^*G \setminus \{0_{T^*G}\}, S^*G)$ is a \mathbb{R}^+ -principal bundle. S^*G admits a canonical contact structure given by the kernel of any one form constructed as the pull-back of the Liouville form on T^*G through a global section of the \mathbb{R}^+ -principal bundle $(\pi, \mathbb{R}^+, T^*G \setminus \{0_{T^*G}\}, S^*G)$. Namely, for every global section $\sigma : S^*G \rightarrow T^*G \setminus \{0_{T^*G}\}$ the one-form $\theta_\sigma = \sigma^*\theta$ determines the same contact structure. Note that $\pi_{\mathbb{R}^+}^*\theta_\sigma = f_\sigma\theta$, where $f_\sigma : T^*G \setminus \{0_{T^*G}\} \rightarrow \mathbb{R}^+$ is a smooth function with the property that $f_\sigma(r\alpha_g) = \frac{1}{r}f_\sigma(\alpha_g)$ for any $r \in \mathbb{R}^+$ and $\alpha_g \in T^*G$. The action by left translations of G on its cotangent bundle induces a free and proper action on the copshere bundle given by

$$g' \cdot \{\alpha_g\} := \{T_{g'g}^*L_{g^{-1}}\alpha_g\},$$

for all $\{\alpha_g\} \in S^*G$ and $g' \in G$. Since it is a proper action which preserves the contact structure, there is always a global section σ such that the action will preserve the associated contact form θ_σ . Then, this action admits an equivariant momentum map defined by

$$\langle J_{sL}(\{\alpha_g\}), \xi \rangle := \theta_\sigma(\{\alpha_g\})(\xi_{S^*G})(\{\alpha_g\}) = f_\sigma(\alpha_g)\alpha_g(\xi_G(g)),$$

where $\{\alpha_g\} \in S^*G$ and $\xi \in \mathfrak{g}$. That is, $J_{sL}(\alpha_g) = f_\sigma(\alpha_g)\alpha_g$, for any $\{\alpha_g\} \in S^*G$. Here we have briefly recalled the construction and some of the properties of the cosphere bundle of a Lie group. For more details the interesting reader is referred to [9], [11], and [30].

Denote by $\text{Diag}\left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu}\right)$ the diagonal product of the π -quotient of the ray orbit of μ and $\frac{G}{K_\mu}$. The quoteint space $S^*(\mathcal{O}_{\mathbb{R}^+\mu})$ is a smooth manifold since the \mathbb{R}^+ -action on $\mathcal{O}_{\mathbb{R}^+\mu}$ is free and proper. The map

$$[g] \in \frac{G}{G_{\mathbb{R}^+\mu}} \longrightarrow Ad_g^*r\mu$$

is a diffeomorphism. Define the following one form on $\text{Diag} \left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu} \right)$

$$(5.8) \quad \eta_{\mathcal{O}_{\mathbb{R}^+\mu}}(\{Ad_g^* r\mu\}, \hat{g})(T_{Ad_g^* r\mu} \pi_{\mathbb{R}^+\mu}(ad_\xi^* Ad_g^* r\mu + r' Ad_g^* r\mu, \hat{\xi}_G(\hat{g}))) := f_\sigma(T_g^* R_{g^{-1}} r\mu) \langle Ad_g^* r\mu, \xi \rangle,$$

for any $(\{Ad_g^* r\mu\}, \hat{g}) \in \text{Diag} \left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu} \right)$ and any tangent vector

$$T_{Ad_g^* r\mu} \pi_{\mathbb{R}^+\mu}(ad_\xi^* Ad_g^* r\mu + r' Ad_g^* r\mu, \hat{\xi}_G(\hat{g})) \in T_{(\{Ad_g^* r\mu\}, \hat{g})} \text{Diag} \left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu} \right).$$

As we will see in the proof of the following Theorem, the diagonal manifold $(\text{Diag} \left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu} \right), \eta_{\mathcal{O}_{\mathbb{R}^+\mu}})$ is a well defined exact contact manifold.

Theorem 5.3. *Let the Lie group G act on its cosphere bundle S^*G by the lift of left translations on itself. Suppose μ is an element of the dual of its Lie algebra with kernel group K_μ and the property that $\ker \mu + \mathfrak{g}_\mu = \mathfrak{g}$, where \mathfrak{g}_μ is the isotropy algebra of μ for the coadjoint action. Then the ray reduced space at μ , $(S^*G)_{\mathbb{R}^+\mu}$ is well defined and contactomorphic to $(\text{Diag} \left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu} \right), \eta_{\mathcal{O}_{\mathbb{R}^+\mu}})$, where $\eta_{\mathcal{O}_{\mathbb{R}^+\mu}}$ is the one form define by (5.8).*

Proof. First note that since the K_μ and \mathbb{R}^+ -actions commute we have the equality $J_{sL}^{-1}(\mathbb{R}^+\mu) = \pi(J_L^{-1}(\mathbb{R}^+\mu))$. Even more the maps,

$$\phi : (S^*G)_{\mathbb{R}^+\mu} \rightarrow \frac{(T^*G)_{\mathbb{R}^+\mu}}{\mathbb{R}^+} \quad \text{and} \quad \bar{J}_{R^+} : \frac{(T^*G)_{\mathbb{R}^+\mu}}{\mathbb{R}^+} \rightarrow \text{Diag} \left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu} \right)$$

defined by $\phi([\alpha_g]) := [\alpha_g]$ and $\bar{J}_{R^+}([\{T_g^* R_{g^{-1}} r\mu\}, \hat{g})] := (\{Ad_g^* r\mu\}, \hat{g})$, for any α_g in $J_L^{-1}(\mathbb{R}^+\mu)$ are diffeomorphisms. Let $\Psi : (S^*G)_{\mathbb{R}^+\mu} \rightarrow \text{Diag} \left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu} \right)$ be the map $\Psi := \bar{J}_{R^+} \circ \phi$. It is obviously a diffeomorphism with inverse given by

$$\Psi^{-1} : \text{Diag} \left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu} \right) \rightarrow (S^*G)_{\mathbb{R}^+\mu}, \quad \Psi^{-1}(\{Ad_g^* r\mu\}, \hat{g}) = [\{T_g^* R_{g^{-1}} r\mu\}],$$

for any $g \in G$ and $r \in \mathbb{R}^+$. Denote by $\eta_{\mathbb{R}^+\mu}$ the reduced contact form of $(S^*G)_{\mathbb{R}^+\mu}$. Then,

$$(5.9) \quad (\Psi^{-1})^*(\eta_{\mathbb{R}^+\mu})(\{Ad_g^* r\mu\}, \hat{g})(T_{Ad_g^* r\mu} \pi_{\mathbb{R}^+\mu}(ad_\xi^* Ad_g^* r\mu + r' Ad_g^* r\mu, \hat{\xi}_G(\hat{g}))) = \\ (\pi_{\mathbb{R}^+\mu}^s)^*(\{T_g^* R_{g^{-1}} r\mu\}) \left(\frac{d}{dt} \Big|_{t=0} \pi_{\mathbb{R}^+}(Ad_{g \exp t\xi}^* e^{tr'} Ad_g^* r\mu, g \cdot \widehat{\exp t\xi}) \right) = \\ (\pi_{\mathbb{R}^+\mu}^s)^*(\{T_g^* R_{g^{-1}} r\mu\}) \left(T_{T_g^* R_{g^{-1}} r\mu} \pi_{\mathbb{R}^+}(X^\xi(T_g^* R_{g^{-1}} r\mu)) \right) = \\ \theta_\sigma(\{T_g^* R_{g^{-1}} r\mu\}) \left(T_{T_g^* R_{g^{-1}} r\mu} \pi_{\mathbb{R}^+}(X^\xi(T_g^* R_{g^{-1}} r\mu)) \right) = \\ (\pi_{\mathbb{R}^+}^* \theta)(T_{T_g^* R_{g^{-1}} r\mu})(X^\xi(T_g^* R_{g^{-1}} r\mu)) = f_\sigma(T_g^* R_{g^{-1}} r\mu) \theta(T_g^* R_{g^{-1}} r\mu)(X^\xi(T_g^* R_{g^{-1}} r\mu)) \\ = f_\sigma(T_g^* R_{g^{-1}} r\mu) J_R^\xi(T_g^* R_{g^{-1}} r\mu) = f_\sigma(T_g^* R_{g^{-1}} r\mu) \langle Ad_g^* r\mu, \xi \rangle = \\ \eta_{\mathcal{O}_{\mathbb{R}^+\mu}}(\{Ad_g^* r\mu\}, \hat{g})(T_{Ad_g^* r\mu} \pi_{\mathbb{R}^+\mu}(ad_\xi^* Ad_g^* r\mu + r' Ad_g^* r\mu, \hat{\xi}_G(\hat{g}))),$$

for all $\xi \in \mathfrak{g}$ and $g \in G$. Hence, $\eta_{\mathcal{O}_{\mathbb{R}^+\mu}}$ is a contact form and Ψ the required contactomorphism. \square

Corollary 5.2. *In the hypothesis of Theorem 5.3, the contact form $\eta_{\mathcal{O}_{\mathbb{R}^+\mu}}$ defined by 5.8 is G -invariant with respect to the following action*

$$g_1 \cdot (\{Ad_g^* r\mu\}, \hat{g}) := \left(\{Ad_{g_1^{-1}}^* Ad_g^* r\mu\}, \widehat{gg_1^{-1}} \right),$$

for each g_1 in G and $(Ad_g^* r\mu, \hat{g}) \in \text{Diag} \left(S^*(\mathcal{O}_{\mathbb{R}^+\mu}) \times \frac{G}{K_\mu} \right)$.

6. RAY QUOTIENTS OF KÄHLER AND SASAKIAN-EINSTEIN MANIFOLDS

In this section we will study the behavior of Kähler-Einstein metrics of positive Ricci curvature with respect to symmetries. Namely, in the hypothesis of Theorem 2.2 and using techniques developed in [12] and [13] by A. Futaki, if M is a Fano manifold and ω represents its first Chern class we will show how to compute the Ricci form of the reduced space in terms of the reduced Kähler form $\omega_{\mathbb{R}^+\mu}$ and data on $J^{-1}(\mathbb{R}^+\mu)$ and the kernel group K_μ . As a corollary we will obtain that if M is a Fano manifold and the symplectic ray reduction of Theorem 2.1 can be performed, then the ray reduced symplectic manifold $M_{\mathbb{R}^+\mu}$ will also be Fano. Even more, if M is a compact Kähler-Einstein manifold of positive Ricci curvature, then $M_{\mathbb{R}^+\mu}$ is Einstein if and only if the norm of a certain multi vector field defined using the kernel algebra \mathfrak{k}_μ and the algebra \mathfrak{m} defined in (2.3) is constant on $J^{-1}(\mathbb{R}^+\mu)$.

Recall that the Ricci form ρ of a compact Kähler manifold (M, g, ω) is a real closed $(1, 1)$ -form whose class in the de Rham cohomology group $H_{DR}^2(M)$ defines the first Chern class of the manifold. Suppose that the Kähler form ω represents the first Chern class of M . Then, applying the local $i\partial\bar{\partial}$ -Lemma (see, for instance [26]), we obtain that there is a smooth real function f such that $\rho - \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}f$. If the compact Lie group G acts on M by holomorphic isometries, then there is always an associated equivariant momentum map $J : M \rightarrow \mathfrak{g}^*$. We will now recall its construction.

By Theorem 2.4.3 in [12] there is an isomorphism between the complex Lie algebra of holomorphic vector fields on M and the set of all complex-valued functions u satisfying $\Delta_f u - u = 0$. This isomorphism is given by $u \mapsto \text{grad } u$. Here, Δ_f is the differential operator given by

$$u \mapsto \Delta u - \nabla^i u \nabla_i f = \Delta u - g^{i\bar{j}} \frac{\partial u}{\partial \bar{z}_j} \partial_{z_i} f = \Delta u - \text{grad } u(f),$$

with Δ the complex Laplacian, ∇ the covariant derivative associated to g and $(z_i)_i$ local holomorphic coordinates. Then, the infinitesimal isometries associated to the elements of the Lie algebra \mathfrak{g} embed in the space of holomorphic vector fields on M as follows: assign to each $\xi \in \mathfrak{g}$ the holomorphic vector field $\xi'_M := \frac{1}{2}(\xi_M - \sqrt{-1}\mathcal{C}_g \xi_M)$. \mathcal{C} denotes the complex structure of (M, g) . In other words, all the infinitesimal isometries are real holomorphic vector fields. Therefore, there is a smooth complex function $u_{\xi'_M}$ with $\text{grad } u_{\xi'_M} = \xi'_M$.

Lemma 6.1. *For every ξ element of the Lie algebra \mathfrak{g} , the above defined function $u_{\xi'_M}$ is purely imaginary.*

Proof. Since G acts by holomorphic isometries, $\xi_M(f) = 0$ and we have

$$\Delta_f u_{\xi'_M} = \Delta u_{\xi'_M} - \xi'_M(f) = \Delta u_{\xi'_M} + \frac{\sqrt{-1}}{2} \mathcal{C}(\xi_M)f \text{ and } \Delta_f \bar{u}_{\xi'_M} = \Delta \bar{u}_{\xi'_M} - \frac{\sqrt{-1}}{2} \mathcal{C}(\xi_M)f.$$

Using the fact that $\Delta_f u_{\xi'_M} - u_{\xi'_M} = 0$ it follows that

$$(6.1) \quad \Delta(u_{\xi'_M} + u_{\bar{\xi}'_M}) = u_{\xi'_M} + \bar{u}_{\xi'_M}.$$

On the other hand, it is well known that on a complex connected Riemannian manifold, if X , the gradient of a function u is a holomorphic vector field, then it is a Killing vector field if and only if $u + \bar{u}$ is constant. In particular, if u is purely imaginary, then the real part of X is a Killing vector field. For a proof of this, see for instance [7]. Applying this to $X = \text{grad}(u_{\xi'_M} + \bar{u}_{\xi'_M})$ we obtain that $u_{\xi'_M} + \bar{u}_{\xi'_M}$ is a constant function. Hence, (6.1) implies that $u_{\xi'_M} + \bar{u}_{\xi'_M} = 0$. \square

Proposition 6.1. *Let M be a compact complex manifold of positive first Chern class and dimension n . Choose any Kähler metric g which represents the first Chern class and suppose the Lie group G acts on (M, g) by holomorphic isometries. Then the map $J : M \rightarrow \mathfrak{g}^*$, $\langle J(x), \xi \rangle := \frac{\sqrt{-1}}{2\pi} u_{\xi'_M}$ defines an equivariant momentum map for the action of G on M .*

Proof. In a local holomorphic coordinate system (z_1, \dots, z_n) , the Kähler form associated to g is given by $\omega = \frac{\sqrt{-1}}{2\pi} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$. Then, for any ξ in \mathfrak{g} ,

$$i_{\xi'_M} \omega = i_{\text{grad } u_{\xi'_M}} \omega = \frac{i}{2\pi} g^{\alpha\bar{\beta}} \nabla_{\bar{\beta}} u_{\xi'_M} g_{\alpha\bar{\gamma}} d\bar{z}^\gamma = \bar{\partial} J^\xi.$$

and

$$i_{\xi_M} \omega = i_{(\xi'_M + \bar{\xi}'_M)} \omega_g = i_{\xi'_M} \omega + \overline{i_{\xi'_M} \omega} = dJ^\xi,$$

proving thus that J is a momentum map. To show the equivariance of J , fix $g \in G$ and $\xi \in \mathfrak{g}$. Observe that the G -action commutes with the operator Δ_f and that for any vector field Y of type $(0, 1)$ we have

$$\begin{aligned} \omega(\text{grad}(g^* u_{\xi'_M}), Y) &= Y(g^* J^\xi) = (g_* Y) J^\xi = \omega(\text{grad } u_{\xi'_M}, g_* Y) = \\ \omega(g_*^{-1} \xi'_M, Y) &= \omega((ad_{g^{-1}} \xi)'_M, Y) = \omega(\text{grad } u_{(ad_{g^{-1}} \xi)'_M}, Y). \end{aligned}$$

Hence, J is also G -equivariant. \square

Assume that the hypothesis of Theorem 2.2 are verified for a momentum value μ . Choose $\{\xi_i\}_{i=1,k}$ and $\{\eta_i\}_{i=1,m}$ basis of \mathfrak{k}_μ and \mathfrak{m} such that the associated infinitesimal isometries form an orthogonal frame of the vertical distribution of $\pi_{\mathbb{R}+\mu} : J^{-1}(\mathbb{R}^+\mu) \rightarrow M_{\mathbb{R}+\mu}$ and of \mathfrak{m}_M respectively. Recall that \mathfrak{m}_M is the space defined in the decomposition (2.6). Denote by $\xi' \wedge \eta'$ the multi vector $\xi'_{1M} \wedge \dots \wedge \xi'_{kM} \wedge \eta'_{1M} \wedge \dots \wedge \eta'_{mM}$. We are now ready to state the main theorem of this section.

Theorem 6.1. *Let (M, g, ω) be a Fano Kähler manifold with ω representing its first Chern class. Let G be a Lie group acting on M by holomorphic isometries. Suppose that μ is an element of the dual of the Lie algebra of G such that the ray reduced space is a well defined Kähler orbifold $(M_{\mathbb{R}+\mu}, \omega_{\mathbb{R}+\mu})$. Assume that the kernel group K_μ is compact. Then, the Ricci form of the ray reduced space is given by*

$$(6.2) \quad \rho_{\mathbb{R}+\mu} = \omega_{\mathbb{R}+\mu} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(f_{\mathbb{R}+\mu} + \log \|\xi' \wedge \eta'\|_{\mathbb{R}+\mu}^2),$$

where $f_{\mathbb{R}+\mu}$ and $\|\xi' \wedge \eta'\|_{\mathbb{R}+\mu}$ are the K_μ -projections of f and the point-wise norm of the multi vector $\xi' \wedge \eta'$. Consequently, $M_{\mathbb{R}+\mu}$ is also Fano.

Proof. First note that $\xi' \wedge \eta'$ is K_μ -invariant. For any $g \in G$ and $x \in M$, $\xi_M(gx) = g_*(ad_{g^{-1}}\xi)_M(x)$. Since the kernel group K_μ is compact, $\det(ad_{g^{-1}}|_{\mathfrak{k}_\mu}) = 1$ and

$$(\xi'_{1M} \wedge \dots \wedge \xi'_{kM})(gx) = (\det(ad_{g^{-1}}|_{\mathfrak{k}_\mu})g_*(\xi'_{1M} \wedge \dots \wedge \xi'_{kM}))(x) = g_*(\xi'_{1M} \wedge \dots \wedge \xi'_{kM})(x).$$

Even more $\det(ad_{g^{-1}}|_{\mathfrak{g}}) = \det(ad_{g^{-1}}|_{\mathfrak{g}_\mu}) \det(ad_{g^{-1}}|_{\mathfrak{m}})$ and since G_μ and G are compact, it follows that $\det(ad_{g^{-1}}|_{\mathfrak{m}}) = 1$. By an argument similar to the one above we obtain that the multi vector $\xi' \wedge \eta'$ is K_μ -invariant.

The action of K_μ being by isometries it is clear that the point wise norm of $\xi' \wedge \eta'$ is also K_μ -invariant. Recall from Theorem 2.2 that we have the following orthogonal decomposition:

$$(6.3) \quad T_x M = \mathcal{V}_x \oplus \mathcal{H}_x \oplus \mathfrak{m}_M(x) \oplus \mathcal{C}(\mathcal{V}_x).$$

\mathcal{V}_x is the vertical space at x of the Riemannian submersion $\pi_{\mathbb{R}^+\mu} : J^{-1}(\mathbb{R}^+\mu) \rightarrow M_{\mathbb{R}^+\mu}$ and it is generated by $\{\xi_{iM}(x)\}_{i=1,k}$. The horizontal space at x is \mathcal{H}_x and $\mathfrak{m}_M(x)$ is invariant with respect to the complex structure \mathcal{C} . Let \mathcal{V} and \mathcal{M} be the distributions defined by $\{\mathcal{V}_x \oplus \mathcal{C}(\mathcal{V}_x)\}_{x \in J^{-1}(\mathbb{R}^+\mu)}$ and $\{\mathfrak{m}_M(x)\}_{x \in J^{-1}(\mathbb{R}^+\mu)}$. Consider the following decompositions

$$\begin{aligned} \mathcal{V} \otimes \mathbb{C} &= \mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1} \\ \mathcal{H} \otimes \mathbb{C} &= \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1} \\ \mathcal{M} \otimes \mathbb{C} &= \mathcal{M}^{1,0} \oplus \mathcal{M}^{0,1}. \end{aligned}$$

Then we have that $i_{\mathbb{R}^+\mu}^* T^{1,0} M = \mathcal{H}^{1,0} \oplus \mathcal{V}^{1,0} \oplus \mathcal{M}^{1,0}$. Denote by ∇^h , ∇^v , ∇^m , and $\nabla^{\mathbb{R}^+\mu}$ the connections induced by the Levi-Civita connection of M on $\mathcal{H}^{1,0}$, $\mathcal{V}^{1,0}$, $\mathcal{M}^{1,0}$, and $i_{\mathbb{R}^+\mu}^* T^{1,0} M$ (or their determinant bundles). Let θ^h , θ^v , θ^m , and $\theta^{\mathbb{R}^+\mu}$ be the connection forms of the above defined connections with respect to the local, orthogonal and K_μ -invariant frames $Y_1 \wedge \dots \wedge Y_s$, $\xi'_{1M} \wedge \dots \wedge \xi'_{kM}$, $\eta'_{1M} \wedge \dots \wedge \eta'_{mM}$, $Y_1 \wedge \dots \wedge Y_s \wedge \xi'_{1M} \wedge \dots \wedge \xi'_{kM} \wedge \eta'_{1M} \wedge \dots \wedge \eta'_{mM}$, respectively. Then, $\theta^{\mathbb{R}^+\mu} = \theta^h + \theta^v + \theta^m$. Extend the connection forms by

$$\begin{aligned} \theta_h^h(Y) &= \theta^h(Y) & \theta_h^h(\xi_M) &= 0 & \theta_h^h(\eta_M) &= 0 & \theta_v^h(Y) &= 0 & \theta_v^h(\xi_M) &= \theta^h(\xi_M) \\ \theta_h^v(Y) &= \theta^v(Y) & \theta_h^v(\xi_M) &= 0 & \theta_h^v(\eta_M) &= 0 & \theta_v^v(Y) &= 0 & \theta_v^v(\xi_M) &= \theta^v(\xi_M) \\ \theta_h^m(Y) &= \theta^m(Y) & \theta_h^m(\xi_M) &= 0 & \theta_h^m(\eta_M) &= 0 & \theta_v^m(Y) &= 0 & \theta_v^m(\xi_M) &= \theta^m(\xi_M) \end{aligned}$$

$$\begin{aligned} \theta_v^h(\eta_M) &= 0 & \theta_m^h(Y) &= 0 & \theta_m^h(\xi_M) &= 0 & \theta_m^h(\eta_M) &= \theta^h(\eta_M) \\ \theta_v^v(\eta_M) &= 0 & \theta_m^v(Y) &= 0 & \theta_m^v(\xi_M) &= 0 & \theta_m^v(\eta_M) &= \theta^v(\eta_M) \\ \theta_v^m(\eta_M) &= 0 & \theta_m^m(Y) &= 0 & \theta_m^m(\xi_M) &= 0 & \theta_m^m(\eta_M) &= \theta^m(\eta_M) \end{aligned},$$

for any $Y \in \mathcal{H}$, $\xi_M \in \mathcal{V}$, and $\eta_M \in \mathcal{M}$. Then $\theta = \theta_h^h + B$, where $B = \theta_v^h + \theta_m^h + \theta_h^v + \theta_v^v + \theta_m^v + \theta_h^m + \theta_m^m$. Finally, let $\theta_{\mathbb{R}^+\mu}$ be the connection form of the fiber bundle $\det T^{1,0} M_{\mathbb{R}^+\mu}$ with respect to the local orthogonal frame $\pi_{\mathbb{R}^+\mu} Y_1 \wedge \dots \wedge \pi_{\mathbb{R}^+\mu} Y_s$.

We want to prove that $(\pi_{\mathbb{R}^+\mu})^* \theta_{\mathbb{R}^+\mu} = \theta_h^h$. First, note that the Levi-Civita connection of $M_{\mathbb{R}^+\mu}$ is given by

$$\nabla_{\hat{X}_1}^{\mathbb{R}^+\mu} \hat{X}_2 = \pi_{\mathbb{R}^+\mu}^*(hor(\nabla_{X_{1h}} X_{2h})),$$

for any \hat{X}_1, \hat{X}_2 vector fields on the quotient. Here hor denotes the horizontal projection and X_{1h}, X_{2h} are the unique sections of the horizontal distribution

which project onto \hat{X}_1 and \hat{X}_2 . Then, we obtain

$$\begin{aligned} (\pi_{\mathbb{R}^+\mu}^* \theta_{\mathbb{R}^+\mu})(X_h) &= \nabla_{\pi_{\mathbb{R}^+\mu}^* X_h}^{\mathbb{R}^+\mu} (\pi_{\mathbb{R}^+\mu}^* (Y_1) \wedge \dots \wedge \pi_{\mathbb{R}^+\mu}^* (Y_k)) = \\ &= \sum_{i=1}^k \pi_{\mathbb{R}^+\mu}^* (Y_1) \wedge \dots \wedge \pi_{\mathbb{R}^+\mu}^* (\text{hor} \nabla_{X_h} Y_i) \wedge \dots \wedge \pi_{\mathbb{R}^+\mu}^* (Y_k) \\ &= \pi_{\mathbb{R}^+\mu}^* (\nabla_{X_h} (Y_1 \wedge \dots \wedge Y_k)) = \pi_{\mathbb{R}^+\mu}^* (\theta_h^h(X_h) Y_1 \wedge \dots \wedge Y_k). \end{aligned}$$

Since the frame is K_μ -invariant it follows that $\theta_h^h(X_h)$ is a K_μ -invariant function on $J^{-1}(\mathbb{R}^+\mu)$, for any horizontal vector field X_h . Therefore, $(\pi_{\mathbb{R}^+\mu}^*)^* \theta_{\mathbb{R}^+\mu} = \theta_h^h$ and

$$\pi_{\mathbb{R}^+\mu}^* \rho_{\omega_{\mathbb{R}^+\mu}} = \frac{\sqrt{-1}}{2\pi} d\pi_{\mathbb{R}^+\mu}^* \theta_{\mathbb{R}^+\mu} = \frac{\sqrt{-1}}{2\pi} d\theta_h^h = \frac{\sqrt{-1}}{2\pi} (d\theta - B) = i_{\mathbb{R}^+\mu}^* \rho_\omega - \frac{\sqrt{-1}}{2\pi} B,$$

where $B := d\theta_v^h + d\theta_m^h + d\theta_h^v + d\theta_v^v + d\theta_m^v + d\theta_h^m + d\theta_v^m + d\theta_m^m$.

Observe that

$$(6.5) \quad d\theta_h^v = d\pi_{\mathbb{R}^+\mu}^* (\partial \log \|\xi'\|_{\mathbb{R}^+\mu}^2) = \pi_{\mathbb{R}^+\mu}^* (\bar{\partial} \partial \log \|\xi'\|_{\mathbb{R}^+\mu}^2).$$

Indeed, fix Y a section of $\mathcal{H}^{1,0}$. Working in holomorphic coordinates it is very easy to see that $\nabla_Y^v \xi'_M = 0$. On the other hand

$$(6.6) \quad \begin{aligned} \nabla_Y^v \xi' &= \sum_{i=1}^k \xi'_{1M} \wedge \dots \wedge \frac{g(\nabla_Y \xi'_{iM}, \bar{\xi}'_{iM})}{\|\xi'_{iM}\|^2} \xi'_{iM} \wedge \dots \wedge \xi'_{kM} = \\ &= \sum_{i=1}^k Y(\log(\|\xi'_{iM}\|^2)) \xi' = Y(\log(\|\xi'\|^2)) \xi'. \end{aligned}$$

Hence formula (6.5) follows. In a similar way we can see that $d\theta_h^m = \pi_{\mathbb{R}^+\mu}^* (\bar{\partial} \partial (\log \|\eta'\|^2))$. Therefore,

$$(6.7) \quad d\theta_h^v + d\theta_h^m = \pi_{\mathbb{R}^+\mu}^* (\bar{\partial} \partial \log \|\xi' \wedge \eta'\|^2).$$

Applying Lemma 7.3.8 in [12], we know that for any γ , section of $\det T^{1,0}M$ and any ξ in \mathfrak{k}_μ , $L_{\xi_M} \gamma = \nabla_{\xi_M} \gamma - (2\pi\sqrt{-1}\Delta J^\xi) \gamma$. In particular, for $\gamma := Y_1 \wedge \dots \wedge Y_s \wedge \xi'_{1M} \wedge \dots \wedge \xi'_{kM} \wedge \eta'_{1M} \wedge \dots \wedge \eta'_{mM}$, along $J^{-1}(\mathbb{R}^+\mu)$ we get $\nabla_{\xi_M} \gamma = L_{\xi_M} \gamma + (2\pi\sqrt{-1}\Delta J^\xi) \gamma = -(\xi'_M f) \gamma$ and $\nabla_{\eta_M} \gamma = -(\eta'_M f) \gamma$. Recall that from the definition of J we have that $u_{\xi'_M} = u_{\eta'_M} = 0$, for all $\xi \in \mathfrak{k}_\mu$ and $\eta \in \mathfrak{m}$. Let $\theta_v := \theta_v^h + \theta_v^v + \theta_v^m$ and $\theta_m := \theta_m^h + \theta_m^v + \theta_m^m$. From the above computations we have that $\theta_v(\xi_M) = -\xi'_M(f)$ and $\theta_m(\eta_M) = -\eta'_M(f) = 0$, for all $\xi \in \mathfrak{k}_\mu$ and all $\eta \in \mathfrak{m}$. Notice that for the last equality we have used the fact that \mathfrak{m}_M is invariant with respect to the complex structure \mathcal{C} .

The definitions of θ_v and θ_m imply that

$$(6.8) \quad \theta_v = -i_{\mathbb{R}^+\mu}^* \partial f + \pi_{\mathbb{R}^+\mu}^* \partial f_{\mathbb{R}^+\mu} \quad d\theta_v = i_{\mathbb{R}^+\mu}^* \partial \bar{\partial} f - \pi_{\mathbb{R}^+\mu}^* \partial \bar{\partial} f_{\mathbb{R}^+\mu},$$

$$(6.9) \quad d\theta_m = 0.$$

From (6.4), (6.7), (6.9), and (6.8), the conclusion of the theorem follows. \square

Theorem 2.2, Proposition 3.2, and Theorem 6.1 entail the following corollary.

Corollary 6.1. *In the hypothesis of Theorem 2.2, suppose M is also Kähler-Einstein of positive Ricci curvature. Then ray reduced space $M_{\mathbb{R}^+\mu}$ is Kähler-Einstein if and only if $\|\xi' \wedge \eta'\|_{\mathbb{R}^+\mu}$ is constant on $J^{-1}(\mathbb{R}^+\mu)$.*

Proof. It is just a matter of definitions. \square

Theorems 2.5 in [4] and 6.1 imply

Corollary 6.2. *In the hypothesis of Theorem 6.1, if M has Ricci curvature strictly bigger than -2 , then so does $M_{\mathbb{R}^+ \mu}$.*

Examples 6.1. We will now show that all the reduced spaces obtained in some examples of [8] are in fact Sasakian-Einstein manifolds. In Example 3.2 of this article we let the torus T^2 act on the sphere S^7 as follows:

$$((e^{\sqrt{-1}t_0}, e^{\sqrt{-1}t_1}), z) \mapsto (e^{-\sqrt{-1}t_0} z_0, e^{\sqrt{-1}t_0} z_1, e^{\sqrt{-1}t_1} z_2, e^{\sqrt{-1}t_1} z_3).$$

Recall that the infinitesimal generator is given by:

$$(r_1, r_2)_{S^7}(z) = r_1(y_0 \partial_{x_0} - x_0 \partial_{y_0}) + r_1(-y_1 \partial_{x_1} + x_1 \partial_{y_1}) \\ + r_2(-y_2 \partial_{x_2} + x_2 \partial_{y_2}) + r_2(-y_3 \partial_{x_3} + x_3 \partial_{y_3}),$$

for any (r_1, r_2) in the Lie algebra of T^2 and the momentum map by $J(z) = \langle (|z_1|^2 - |z_0|^2, |z_2|^2 + |z_3|^2), \cdot \rangle$, for any $z \in S^7$. For $\mu := \langle v, \cdot \rangle$, $v = (1, 1)$ we have that $M_{\mathbb{R}^+ \mu}$ can be diffeomorphically identified with $S^5(\frac{1}{\sqrt{2}}) \setminus \text{pr} \{z \in S^7 \mid |z_0|^2 = \frac{1}{2}\} \simeq S^5(\frac{1}{\sqrt{2}}) \setminus S^1(\frac{1}{\sqrt{2}})$, where $\text{pr} : \mathbb{C}^4 \rightarrow \mathbb{C}^3$, $\text{pr}(z_0, \dots, z_3) = (z_0, z_2, z_3)$. Since the group is commutative the algebra \mathfrak{m} defined in (2.3) can be identified with $\{0\}$ and the multivector field of Corollary 6.1 turns out to be a simple vector field $\xi' = (-r, r)'_{S^7} = (-r, r)_{S^7} - \sqrt{-1}(-r, r)_{S^7}$, with r any non zero real number. A simple calculation shows that $\|(-r, r)'_{S^7}\|(z) = |r|\|z\| = |r|$ for all $z \in J^{-1}(\mathbb{R}^+ \mu)$. Hence $\|(-r, r)'_{S^7}\|$ it is constant on $J^{-1}(\mathbb{R}^+ \mu)$ and $M_{\mathbb{R}^+ \mu}$ is a Sasakian-Einstein manifold.

In Example 3.4 of the same article, a new Sasakian manifold is obtained for μ defined exactly as above and the initial action on S^7 weighted into

$$((e^{it_0}, e^{it_1}), z) \mapsto (e^{it_0 \lambda_0} z_0, e^{it_1 \lambda_1} z_1, z_2, z_3),$$

with λ_0, λ_1 positive constants. This time, the norm of $\xi'(z_0, z_1, z_2, z_3) = (-1, 1)'_{S^7}(z_0, z_1, z_2, z_3)$ equals $\sqrt{2}|\lambda_1|\|z_1\|$ which is not constant on $J^{-1}(\mathbb{R}^+ \mu) = S^7 \cap (\mathbb{C}^* \times A)$ with A is the ellipsoid of equation

$$|z_1|^2 \left(1 + \frac{\lambda_1}{\lambda_0}\right) + |z_2|^2 + |z_3|^2 = 1.$$

Therefore, the reduced Sasakian manifold

$$M_{\mathbb{R}^+ \mu} = \bigcup_{(z_2, z_3) \in \text{pr}(J^{-1}(\mathbb{R}^+ \mu))} S^1(\beta^{-\lambda_0} \alpha^{\lambda_1}) \times \{(z_2, z_3)\}$$

where $\text{pr} : \mathbb{C}^4 \rightarrow \mathbb{C}^2$, $\text{pr}(z_0, \dots, z_3) = (z_2, z_3)$, $\beta = \sqrt{\frac{\lambda_0(1-|z_2|^2-|z_3|^2)}{\lambda_0+\lambda_1}}$, and $\alpha = \sqrt{\frac{\lambda_1(1-|z_2|^2-|z_3|^2)}{\lambda_0+\lambda_1}}$ is not Einstein.

On the other hand, the new contact structure obtained in [9], Example 3.5 is Sasakian-Einstein since the infinitesimal isometries generated by the kernel algebra of μ are independent of the configuration points.

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REFERENCES

- [1] V. I. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluids parfaits*, Ann. Ins. Fourier, Grenoble, **16** (1966), 319–361.
- [2] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, (1971).
- [3] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Math. **203**, Birkhäuser, Boston, Basel, 2002.
- [4] C. P. Boyer, K. Galicki, *On Sasakian-Einstein Geometry*, Internat. J. Math. **11** (2000), no. 7, 873–909.
- [5] C. P. Boyer, K. Galicki, *3-Sasakian Manifolds. Surveys in differential geometry: essays on Einstein manifolds*, Surv. Diff. Geom., VI, Int. Press, Boston, MA, (1999), 123–184.
- [6] R. L. Bryant, *An Introduction to Lie Groups and Symplectic Geometry*. Geometry and quantum field theory (Park City, UT, 1991), 5–181, IAS/Park City Math. Ser., **1**, Amer. Math. Soc., Providence, RI, 1995.
- [7] E. Calabi, *Extremal Kähler Metrics II, Differential Geometry and Complex Analysis* (I. Chavel, H. M. Farkas, eds), Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1985, 95–114.
- [8] O. M. Drăgulete, L. Ornea, *Non-zero contact and Sasakian reduction*, Diff. Geom. Appl., **24** (2006), no. 3, 260–270.
- [9] O. M. Drăgulete, L. Ornea, T. S. Ratiu, *Cosphere Bundle Reduction in Contact Geometry*, J. Symplectic Geom., **1** (2003), 695–714.
- [10] J. J. Duistermaat, J. A. Kolk, *Lie Groups*, (1999), Universitext, Springer-Verlag.
- [11] T. Ekholm, J. B. Etnyre, *Invariants of Knots, Embeddings and Immersions via Contact Geometry*, math.GT/0412517.
- [12] A. Futaki, *Kähler-Einstein Metrics and Integral Invariants*, (1988), Lecture Notes in Mathematics, **1314**, Springer-Verlag.
- [13] A. Futaki, *The Ricci Curvature of Symplectic Quotients of Fano Manifolds*, Tohoku Math. Journ., **39** (1987), 329–339.
- [14] V. Guillemin, S. Sternberg, *Homogenous Quantization and Multiplicities of Group Representations*, J. Funct. Anal., **47** (1982), 344–380.
- [15] Y. Hatakeyama, *Some notes on differentiable manifolds with almost contact structures*, Tôhoku Math. J. **15** (1963), 176–181.
- [16] V. A. Iskovskikh, Y. G. Prokhorov, *Fano varieties*, in Algebraic geometry, V, Encyclop. Math. Sci. **47**, Springer, Berlin (1999), 1–247.
- [17] Y. Kamishima, L. Ornea, *Geometric flow on compact locally conformal Kähler manifolds*, Tohoku Math. J., (2) **57**, (2005), no. 2, 201–221.
- [18] A. A. Kirillov, *Elements of the Theory of Representations*, Grundlehren der mathematischen Wissenschaften, **220**, Springer-Verlag, 1976.
- [19] A. A. Kirillov, *The Orbit Method, I: Geometric Quantization*, Contemporary Mathematics, vol. **145**, (1983), 1–63.
- [20] B. Kostant, *Orbits, symplectic structures and representation theory*, Proc. US-Japan Seminar on Diff. Geom., Kyoto, Nippon Hyronsha, Tokyo, **77** (1965).
- [21] B. Kostant, *On differential geometry and homogenous spaces II*, Proc. N. A. S. U. S. A., **42**, (1956) 354–357.
- [22] R. McLachlan, M. Perlmutter, *Conformal Hamiltonian Systems*, J. Geom. Phys., **39** (2001), 276–300.
- [23] J. E. Marsden, T. S. Ratiu, *Introduction to Mechanics and Symmetry*, second edition (1999), Texts in Applied Mathematics, **17**, Springer-Verlag.
- [24] J. E. Marsden, A. Weinstein, *Reduction of symplectic manifolds with symmetris*, Rep. Math. Phys. **5** (1974), 121–130.
- [25] J. E. Marsden, A. Weinstein, *Comments on the History, Theory, and Applications of the Symplectic Reduction*, Progr. Math., vol. **198**, Birkhäuser Boston, Boston, MA, 2001.
- [26] A. Moroianu, *Lectures on Kähler geometry*, London Mathematical Society Student Texts, **69** (2007), Cambridge University Press.
- [27] L. Ornea, M. Verbitsky, *Immersion theorem for compact Vaisman manifolds*, Math. Ann., **332**, (2005), no. 1, 121–143.

- [28] J-P. Ortega and T.R. Ratiu, *Momentum Maps and Hamiltonian Reduction*, Progress in Mathematics, Volume **222**, Birkhäuser, Boston, 2004.
- [29] R. S. Palais, *On the Existence of Slices for actions of Non-Compact Lie Groups*, Ann. Math., **73** (1961), 295–323.
- [30] T. Ratiu, R. Schmid, The differentiable structure of three remarkable diffeomorphism groups, *Math. Z.*, **177** (1981), 81–100.
- [31] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure*, Tôhoku Math. J. **2** (1960), 459–476.
- [32] S. Sasaki, *Contact structures on Brieskorn manifolds* (lecture Japan Mathematical Society 1975), Shiego Sasaki Selected Papers, Kinokuniya, Tokyo, 349–363.
- [33] J-M. Souriau, *Structure des Systèmes Dynamiques*, Dunod. Paris. English translation by R. H. Cushman and G. M. Tuynman as *Structure of Dynamical Systems. A Symplectic View of Physics*, **149**, Prog. Math. Birkhäuser, 1997.
- [34] P. Stefan, *Accessible sets, orbits, and foliations with singularities*, Proc. Lond. Math. Soc., **29**(1974), 699–713.
- [35] H. Sussmann, *Orbits of families of vector fields and integrability of distributions*, Trans. Amer. Math. Soc., **180** (1973), 171–188.
- [36] C. Willett, *Contact reduction*, Trans. Amer. Math. Soc., **354** (2002), 4245–4260.

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